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ABSTRACT

This is an experimental SMSG mathematics text for junior high school students. Key ideas emphasized are structure of arithmetic from an algebraic viewpoint, the real number system as a progressing development, and metric and non-metric relations in geometry. Chapter topics include why study mathematics, decimal and non-decimal numeration, the natural numbers and zero, factoring and primes, divisibility, unsigned rationals, non-metric geometry, measurement, informal geometry, approximation, the lever, statistics, chance, and finite mathematical systems. (MP)

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EXPERIMENTAL UNITS FOR GRADES SEVEN AND EIGHT

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EXPERIMENTAL UNITS FOR GRADES SEVEN AND EIGHT

Prepared by the SCHOOL MATHEMATICS STUDY GROUP
Under a grant from the NATIONAL SCIENCE FOUNDATION

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INTRODUCTION

As a part of the activity of the writing group of the School Mathematics Study Group, working at Yale University in the summer of 1958, the Committee on Grades 7 and 8 prepared a number of units to be tried out by a large number of teachers in these grades during the academic year 1958-59. A number of the units are on content which has rarely if ever been treated in grades 7 and 8; others are based on the more traditional content but developed from a new point of view and with new emphasis. A Teacher's Guide, prepared for each unit, includes a statement of the purpose of the unit, indication of previous pupil mathematical experiences which are assumed, teaching suggestions and background information for the teacher.

The topics of these units are:

1. An Introductory Unit on "Why Study Mathematics?"
2. Decimal and Non-Decimal Numeration.
3. The Natural Numbers and Zero
4. Factoring and Primes

4a. Divisibility (A Supplementary Unit)

5. Unsigned Rationals
6. Non-Metric Geometry
7. Measurement
8. Informal Geometry I. (Angle Relationships)
9. Informal Geometry II. (Congruent Triangles, Perpendicular Bisectors, Parallelograms, Theorem of Pythagoras)
10. Approximation
11. Mathematics in Science -- The Lever (A Supplementary Unit)
12. Statistics - A Unit on Mathematics in the Social Studies
13. Chance
14. Finite Mathematical Systems (A Supplementary Unit)

The Committee makes no recommendations about which of the two grades 7 and 8, would be more suitable for teaching these units. Also, the order of teaching the units is not of great importance, but there are some exceptions. For example, it would be preferable that Units 2, 3, 4, and 5 be in sequence as numbered. Also, it would be preferable that the four units on geometry (6, 7, 8, and 9) be taught in that order. Units 4a, 11, and 14 are labelled as supplementary. The Committee does not propose that these units be used with all students in a class.

The Committee has made free use of materials prepared at the University of Maryland Mathematics Project (Junior High School). In some cases (Units 3, 4, and 14) the Maryland units have been used with only slight modification.

It must be understood that all of the units are exploratory and that the Committee for Grades 7 and 8 has made no attempt to outline a complete mathematics sequence for these grades nor to identify any particular topics that must be included in the mathematics of these grades. Decisions like these will be based on teaching experience with the new materials during the academic year 1958-59.

NOTE: Unit 7 is not included in this volume because of delays in editing and printing.

UNIT I

I. WHAT IS MATHEMATICS AND WHY YOU NEED TO KNOW IT

> "Once, on a train, I fell into conversation with the man next to me. He asked me what kind of work I do. I told him I was a mathematician. He exclaimed, 'you are! Don't you get tired of adding columns of figures all day long?' I had to explain to him that adding numbers can be done by a machine. My job is mainly logical reasoning."

What is this mathematics that so many people are talking about these days? Is it counting and computing? Is it measuring and drawing? Is it a language that uses symbols, like a code? No, mathematics is much more than this. Mathematics is a way of thinking, a way of reasoning. It is a science of reasoning called deductive reasoning.

Let us take an example of reasoning. Suppose there are thirty pupils in your classroom. How can you prove that there are at least two pupils who have birthdays during the same month? To prove this statement you don't need to know the birthdays of the pupils in your room. Instead, you reason like this. Imagine 12 boxes, one for each month of the year. Imagine also that your teacher writes each pupil's name on a slip of paper, then puts each slip into the proper box according to the pupil's birthday. If no box has more than one name, then there could not be more than 12 names. Since there are 30 names then at least one box must contain more than one name.

This is what a mathematician does. He proves statements using logical reasoning. Notice exactly what you proved. You proved that if there are 30 pupils in your room then there are at least two pupils who were born the same month. It is not the mathematician's job to find out whether there are 30 pupils in your room. This can be found out by observation; the teacher can count the pupils. The mathematician's business is the "if -- then" statement. By reasoning he tries to prove that if something is true, then something else must be true.

In arithmetic you have learned how to prove simple statements about numbers. You can tell, without seeing the bags of peanuts and without counting, that if you have 7 bags with 5 peanuts in each bag, then you have 35 peanuts altogether. You can solve more difficult problems by reasoning instead of calculation.

Exercise.

1. What is the least number of students that could be enrolled in a school so that one could be sure there are at least two students with the same birthdate?

2. What is the largest number of students that could be enrolled in a school so that one could be sure that no two students have the same birthdate? 364? 365? Less than 364?

3. If there are 12 movie houses in a certain town, how many people have to go to the show before you can be sure that there will be at least two people in one show?

4. If there are five movie houses in a town what is the smallest number of people that would have to go to the movies before you can be sure that at least two people will see the same show?

5. What is the least number of people that could go to the movie houses so that you could be sure that no two people see the same show?

6. If 8 candy bars are to be divided among 5 boys, how many boys can receive three candy bars if each boy is to receive at least one bar?

7. A 200 pound man and his two sons each weighing 100 pounds want to cross a river. If they have only one boat which can safely carry only 200 pounds, how can they cross the river?

8. A farmer wants to take a goose, a fox, and a bag of corn across a river. If left alone the fox will eat the goose or the goose will eat the corn. If he has only one boat large enough to carry him and one of the others, how does he cross the river?

BRAINBUSTER

9. Three cannibals and three missionaries want to cross a river. They must share a boat which is large enough to

carry only two people. At no time may the cannibals outnumber the missionaries, but the missionaries may outnumber the cannibals. If they work together, how can they all cross the river?

BRAINBUSTER

10. Eight marbles all have the same size, color, and shape. Only one marble differs in weight. Using a balance scale, can you find the heavy marble *if* you make only two weighings?

One way to solve problems is to try all possible ways to see if any of them works. This method is called the *experimental method*. When you use this method you experiment like a scientist in a laboratory. In many problems such an experimental method works. Can you think of some problems that have been solved in this way? A good example is the story of how Edison invented the electric lamp. Try to find out how Edison developed some of his inventions.

Of course the part of mathematics which you know best right now is arithmetic. Even in arithmetic you can obtain results by deductive reasoning which would be hard to obtain by ordinary calculation. When Karl Friedrich Gauss was about 10 years old, his teacher wanted to keep the class quiet for a while. He told the children to add all the numbers from 1 to 100. In about two minutes Gauss was up to mischief again. The teacher asked him why he wasn't working on the problem. He replied "I've done it already!" "Impossible!" exclaimed the teacher. "I'ts easy," answered Gauss. "I wrote

$1 + 2 + 3 \ldots + 100$ and then I reversed the numbers

$\underline{100 + 99 + 98 \ldots + 1}$ and then I added each pair of numbers

$101 + 101 + 101 \ldots + 101$

When I add, I get a hundred 101's, which gives me $100 \cdot 101 = 10{,}100$. But I used each number twice, so I divided my answer by 2."

Answer is $\frac{10{,}100}{2} = 5{,}050$

Who was Karl Gauss? Where did he live? When did he live? What did he do for a living?

Exercise.

1. Add all the digits from 1 to 5 (1 + 2 + 3 + 4 + 5) using Gauss' method. Try to discover the "short" method used by Gauss.

2. Add 0 + 2 + 4 + 6 + 8. Can you do this mentally? How many digits are there in our problem? What is the sum of the first and last digit? 8 + 0 = ? Find the product 5 · 8. Take half of this answer. 1/2 of 40 = 20. Is this the correct sum?

3. Add 1 + 2 + 3 using Gauss' short method.

4. Add 2 + 4 + 6. Does the short method work?

5. Add the numbers 0 + 1 + 2 + 3 ... + 8. Does the short method work?

   (three dots written . . . mean "and so on." a dot is used for each word.)

6. Add the odd numbers from 1 to 15. 1 + 3 + 5 ... + 15.

7. Add the even numbers up to 20. Start with 0 + 2 + 4 + 6 ... + 20.

8. Add the numbers 1 + 2 + 3 ... 25.

9. Add the odd numbers 7 + 9 + 11 ... + 17.

10. Add the even numbers from 22 to 30. 22 + 24 + 26 ... + 30.

11. Add all the numbers from 0 to 50. 0 + 1 + 2 + 3 + 4 ... + 50.

BRAINBUSTER

12. Add all the numbers from 1 to 200. 1 + 2 + 3 + 4 . . . 200. Then add all the numbers from 0 to 200. 0 + 1 + 2 + 3 + 4 ... 200. Are the answers for these two problems the same? Why? Will this method work if we start with 0? If we start with 1? Will this method work for a number series if we start at some place other than 0 or 1? Can you determine why?

A mathematician reasons about all sorts of things. He thinks about numbers and shapes. He tries to figure out the lifting force on an airplane wing. He tries to predict how many automobiles will be sold next year. Everywhere you look you find things to wonder about. If you are full of curiosity you will enjoy being the detective in a life-long mystery story.

The physicist and the engineer make experiments to find out the laws of the flow of air. Mathematicians start from the laws which engineers have discovered by observation. Mathematicians try to find out, by reasoning, how air will flow around a wing of a certain shape if the airplane is moving with a certain speed. They try to predict the lifting force on the wing. The engineer then makes a model airplane and tests it in a wind tunnel. If the mathematician's prediction works, then the engineer may use it to design a new airplane. If the mathematician's prediction does not work he will have to change the "if" part of his reasoning and try again. Nowadays, it takes 100,000 hours of paper work before an engineer can begin the planning work for designing a new airplane.

When Walt Disney considered setting up Disneyland, he had many questions. He wanted to know how big it should be, where it should be, what admission price should be charged, what special facilities he should have for the Fourth of July and other holidays. He didn't want to start building Disneyland at a cost of $17,000,000 until he knew the answers to these questions. No one likes to take a chance on losing that much money. Disney went to the Stanford Research Institute where there is a group of mathematically trained people who are specialists on applying mathematical reasoning to business problems. They gave him the answers he needed to start building.

One topic of interest to mathematicians is probability or chance. Probability is important in many games such as a game with cards. Probability is also important in business, in government, and in science. For example, a mathematician uses probability to predict how many people will be killed in automobile accidents. Government experts predict how much food will be produced by farmers even before the grain is planted. A scientist will predict how many radioactive molecules will fall on your yard tonight. The weatherman uses probability to forecast tomorrow's weather. By using probability with agricultural experiments scientists obtained hybrid corn and rust resistant wheat.

Most mathematicians are not mainly interested in airplanes or farming or business or government. They are mostly interested in the reasoning which is used in solving the problems which arise in these fields. They feel about their work more as a painter or a composer or a poet. Mathematicians find beauty and fascination in the new ideas they discover. You may say that mathematicians work for the fun of it. But the rest of the world pays them to enjoy themselves because they find the results useful. Sometimes a mathematician works on a practical problem. Often his work turns out to be useful only after he has passed away.

Exercise.

To see how probability works, imagine that you toss two pennies. There are 4 possible ways that the coins can come up:

First Penny	H	H	T	T
Second Penny	H	T	H	T

We are using H to represent heads and T to represent tails. HH describes the event of 2 coins coming up heads. We say then that the possibility of tossing 2 heads with 2 coins is 1 out of 4 or 1/4. We cannot predict what will happen in any one toss, but we can predict that if the two coins are tossed 100,000 times then it is likely that two heads will come up about 1/4 of the time.

1. What is the probability of drawing the ace of spades from a deck of cards?
2. What is the probability of drawing any ace from a full deck of cards?
3. What is the probability of throwing a die (one of a pair of dice) and having two dots come up?

BRAINBUSTERS

4. What is the probability of throwing a pair of ones (one dot on each die) with one pair of dice?

5. What is the probability of having 3 heads come up if 3 pennies are tossed?

Why you need to know mathematics

You are living in a world which is changing very rapidly. Scientists are making medical discoveries so that people live longer. Scientists and engineers are finding new things to make and do each year.

If you want to know how much our world has changed, ask your parents and grandparents what life was like when they were in school. How did their way of life compare with ours? You will probably live through more changes and face more new kinds of problems than either your parents or grandparents.

No one knows what these changes will be, or what things you will need to know when you grow up. In fact, you will need to learn very many things which haven't as yet been discovered. Therefore, you need to know, more than anything else, how to get new knowledge. Since any job you get will change very rapidly, your employer will not be so much interested in what you know as much as how will you know how to find things out.

There are two main ways of learning about the world around you. One is the method of observing and experimenting. This is called the inductive method. This is the main method used by the scientists. The other is the method of logical reasoning. This is the deductive method.

The scientist, engineer, and mathematician use both of these methods. You will understand the world around you better if you know how to use these methods.

You should also learn as much as possible about mathematics in other jobs because an ever-growing number of jobs demand more and more mathematical training. As an example take the departments of the government.

Many departments of the government use mathematics in their daily work. The government and private industry employ many mathematicians and mathematically trained people in physics, chemistry, biology, engineering, economics and psychology. There are now about 7,000 professional mathematicians in the United States. The computing machines now on order would require more than twice that number of mathematicians. Even if you don't use mathematics yourself in whatever job you get, and there will be fewer jobs like that as time goes on, you will also have to know how mathematics is used in every phase of modern life in order to be a good citizen. For example, most of the bills before Congress today involve mathematics or science in some way.

Finally when you grow up, you will have more leisure than people have had before. You can choose between a dull, empty life or an exciting, full life. When you learn a way of creating beauty yourself and appreciating the creation of others, you will enjoy an exciting life. By learning mathematics, what it is and how it works, you will be prepared for a successful and creative life.

UNIT II

NUMERATION

HOW OTHER PEOPLES WROTE NUMERALS

What does the numeral 358 mean?

The 3 means 3 hundreds, the 5 means 5 tens, and the 8 means 8 ones. 358 can also be written as 3 hundreds + 5 tens + 8 ones. Adding these parts, the whole numeral means three hundred fifty eight.

People did not always use this system for writing numerals. When primitive people kept a record of a number, they often did it by making scratches in the dirt or on a stone, cutting notches in a stick, or tying knots in a rope. They kept track of their sheep or other animals by placing pebbles in a pile, one for each animal. Later they knew that sheep were missing if there was not one sheep for each pebble. We do much the same thing when we count the votes in a class election, making one mark for each vote, like this: ~~||||~~ ||| . The important thing about these ways of recording numbers is that one mark stands for one object, and there are as many marks as there are objects. They did not have any marks which stood for several things, such as our "8" to stand for "||||||||" or the Roman "V" to stand for "|||||."

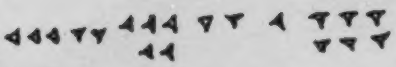
Much later, people began to use a single symbol to stand for several objects. The Egyptians used symbols of this kind. They had different symbols for 1, 10, 100, 1000, 10,000, 100,000 and 1,000,000.

Their symbols are shown below.

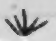
Number	Symbol	Resembles
1	𓏺	Stroke
10	𓎆	Heel bone
100	𓍢	Coiled rope
1000	𓆼	Lotus flower
10,000	𓂭	Bent reed
100,000	𓆐	Burbot fish
1,000,000	𓁨	Astonished man

They repeated the symbol to show other numbers. Their symbol for 100 was "𓍢", resembling a coiled rope, and they wrote seven hundred as "𓍢𓍢𓍢𓍢𓍢𓍢𓍢". The order in which the symbols were written was not important, and hundreds could be written either before or after thousands. They could write "35" either as 𓎆𓎆𓎆𓏺𓏺𓏺𓏺𓏺 or 𓏺𓏺𓏺𓏺𓏺𓎆𓎆𓎆. "2341" was written 𓆼𓆼𓍢𓍢𓍢𓎆𓎆𓎆𓎆𓏺 or 𓍢𓍢𓍢𓆼𓆼𓎆𓎆𓎆𓎆𓏺 or 𓏺𓎆𓎆𓎆𓎆𓍢𓍢𓍢𓆼𓆼. Notice that the numerical value of an Egyptian numeral is the sum of the numbers represented by the individual numerals.

The Babylonians, who lived long ago in the part of Asia which we call the Middle East, had an interesting way of writing numerals. They did their writing on clay tablets with a wedge-like instrument called a stylus. Their symbol for 1 was 𒁹 and their symbol for 10 was 𒌋. They repeated and combined these symbols to write the numbers up to 59. For example, "𒌋𒁹𒁹" meant twelve, and "𒌋𒌋𒌋𒌋𒁹𒁹𒁹𒁹𒁹" meant forty-five. To write numbers larger than 59 they used the same symbols, but the position in which they were written changed their value. They used the number sixty in the same way that we use ten. If we write 452, it means (4 x 10 x 10) + (5 x 10) + (2 x 1). When the Babylonians wrote

 it meant (32 x 60 x 60) + (52 x 60) + (16 x 1). Since they had no symbol for zero until about 200 B. C., they sometimes left empty spaces in their numerals where we would write zeros. It is thought that our way of dividing an hour into sixty minutes and a minute into sixty seconds is related to the way the Babylonians used sixty in writing large numbers.

1. How would the Babylonian numeral above be written in our numerals?
2. Since order made no difference in the meaning of the Egyptian numerals, in what other ways than those shown might they have written 2341? How might they have written 37? 1,111,111?

You have often seen Roman numerals. This system used strokes, which many historians believe were pictures of fingers, for the numbers from one to four, like this: | || ||| |||| . Then they used a hand, for five, like this:  . Gradually they began to leave out some of these marks and wrote five this way: V. They put two hands together for ten, X , which later became X. To write other numbers they combined these by addition. For example, XVII = 10 + 5 + 1 + 1 = 17; CLXIIII = 100 + 50 + 10 + 1 + 1 + 1 + 1 = 164

3. Write the following in Roman numerals:

   (a) 51 (b) 75 (c) 160 (d) 512 (e) 1958

4. Write the following in ordinary Hindu-Arabic numerals:

   (a) VIII (b) CV (c) LXI (d) MCCL (e) DIII

Much later, the Romans began to use subtraction to write some numbers, and wrote four as IV, nine as IX, and forty as XL. As you know, L means fifty, C means 100, D means 500 and M means 1000. Sometimes they wrote a bar over X, C, and M, and that multiplied the value by

1000. $\overline{X}$ means 10,000, $\overline{C}$ means 100,000, and $\overline{M}$ means 1,000,000.

5. Look up the Greek, Hebrew, and Chinese numerals. Make charts of these notations for display in your classroom.
6. How would the sum of 3 and 5 have been written by a Greek child? How would he write 30 + 50? How does this compare with the usual way of solving these two problems?
7. Look up Al-Khwarizmi, Gerbert (Pope Sylvester II), Adelhard of Bath, and Leonardo of Pisa (Fibonacci).
8. Find out where our word "digit" comes from.
9. Choose a number and write it in as many different kinds of numerals as you can.

OUR DECIMAL NUMERALS

Our way of writing numerals was invented in India, and brought to Europe by Arabs. For this reason they are called Hindu-Arabic numerals, although the symbols the Arabs use now are different from our symbols. The important characteristics about this system of writing numerals are:

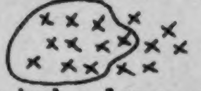
(1) each symbol is the name of a number;

(2) the position of a symbol in a numeral tells the size of a group; and

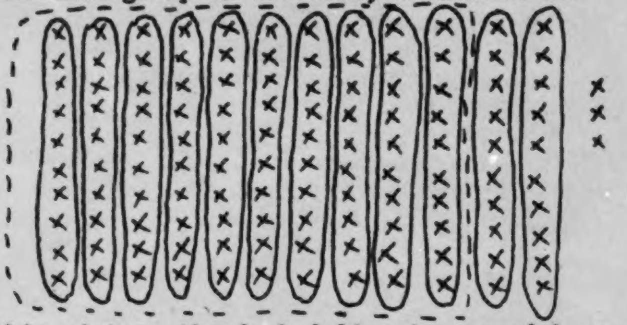
(3) there is a symbol for zero, which is used to fill places which would otherwise be empty and might lead to misunderstanding.

For example, 3 is the numeral for a set of three things. 30 is the numeral for 3 collections of ten things each, or thirty things. 300 is the numeral for 3 collections of one hundred things each, or three hundred things. Addition, multiplication, subtraction, and division are usually easier to perform with our system than with other systems, as you will see if you try to add or multiply using Egyptian, Roman, or Babylonian numerals.

The system we use for writing numerals is called a decimal system. The word "decimal" comes from a Latin word which means ten. We can write the numeral for any number, however large or small, by using just ten number symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. This is possible, because the way we write numerals makes use of place value; that is, the position in which a symbol is written, as well as the symbol, determines the number for which it stands. The 3 in 358 stands for three hundreds, the 5 for 5 tens or fifty, and the 8 for eight. When we count a group of objects, we usually group them in tens. For example, in the x's below a line is drawn around a group of ten x's. So we say there are 16 x's -- one group of ten, and six more.



Suppose we had a large group of x's. We can drawn lines around groups of ten until fewer than ten are left. Then we can draw another line around each group of ten tens, or a hundred.



In this picture the dashed line is around 1 group of 10 tens or 1 hundred; there are also 2 tens, and 3. So there are 1x(10x10) + 2x10 + 3x1, or 123.

10. Draw twenty-seven x's, and draw lines around groups of ten to show what 27 means.

11. Draw a group of x's in which all of the x's are in groups of 10. Write the numeral for this group.

12. Write the meaning of these numerals in the way shown for 48.

48 = (4x10) + (8x1)

(a) 354

(b) 6421

(c) 709

(d) 320

(e) 2000

13. Copy and complete the following multiplication table, using Roman numerals.

	I	V	X	L	C	D	M
I							
V							
X							
L							
C							
D							
M							

When we add numbers we make use of groups of ten also. When we say "How much is 9 + 7?", we mean "How many tens and ones?" So we do not say, "9+7=5+11," although that would be correct. We can regroup as follows:

9+7 =		9+7 =
9+(1+6) =	or	(6+3)+7 =
(9+1)+6 =		6+(3+7) =
10+6		
16		6+10 =
		16

We say our system of writing numerals has the base ten.

Starting at ones' place, each place to the left is given a value 10 times as large as the place before. The places from right to left have the values shown below.

10x10x10x10x10	10x10x10x10	10x10x10	10x10	10	1

Often we write these values more briefly by using a small numeral to the right and above the 10. This numeral shows how many 10's are multiplied together. In this way, the values of the places are written:

10^5	10^4	10^3	10^2	10^1	1

and are read

10 to the fifth power, 10 to the fourth power, 10 cubed, 10 squared, 10 to the first power, 1.

Numerals used as the 5, 4, 3, 2, and 1 are used above are called "exponents" and the numeral with which they are used (in this case, 10) is called the base. The number represented by the entire expression, such as 10^4, is called a power. 4^3 is a short way to write 4x4x4.

14. What is a short way to write 3x3x3x3?

15. What is the meaning of 5^4?

16. What number is represented by 4^3?

17. Which represents the larger number: 4^3 or 3^4?

The meaning of 352 may now be written using exponents, like this: $(3\times10^2) + (5\times10^1) + (2\times1)$.

18. Write the numbers below using exponents.

(a) 468 (b) 5324 (c) 7062 (d) 59,120

Probably the reason that we use a numeral system with ten as base is that people have ten fingers, and when primitive men began to count their possessions they counted on their fingers. This accounts for the fact that the ten symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are called "digits." We speak of these symbols as digits when we wish to refer to them apart from the longer numerals in which they are used. For example, the "digits" in 458 are 4, 5, and 8. The Celts, who lived in Europe more than 2,000 years ago, used twenty as base, and so did the Mayans in Central America. Can you think of a reason? What special word do we sometimes use for twenty? If Martians had a different number of fingers they might use some other number as the base of their numerals. Let us see how systems with bases other than ten work.

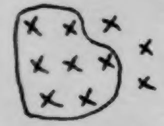
19. Look up the French words for "eighty" and "ninety." Do you know an English word which indicates that some people used to count by twenties? (Think of Lincoln's Gettysburg Address.)

NUMERALS IN BASE SEVEN

Suppose we have a system with seven as base. What symbols shall we have?

0, 1, 2, 3, 4, 5, 6.

There are seven digits or symbols needed for a system of numerals to the base seven just as there were ten needed for base ten. Let us group the x's below so as to write the number of x's in base seven.



We draw a line around seven x's, and see that there is 1 group of seven, and 3 more. There are 13_{seven}. We write the "seven" to show what base we are using. "13_{seven}" means 1 group of seven and 3 ones. Up to 66_{seven}, you may think of the numbers in terms of weeks and days. When no base is written, we understand the numeral is written in base ten.

20. Draw x's and group them with lines to show the meaning of

    (a) 25_{seven} (b) 32_{seven} (c) 40_{seven} (d) 123_{seven}

    Then write each of the above in decimal notation.

21. Numerals in base seven can be written out like this:

    $246_{seven} = (2 \text{ x seven}^2) + (4 \text{ x seven}^1) + (6 \text{ x one}) =$

    $(2 \text{ x } 7 \text{ x } 7) + (4 \text{ x } 7) + (6 \text{ x } 1) = ?$

    How would you write this numeral in the usual decimal notation?

    Write these numerals with exponents in the above way:

    (a) 56_{seven} (b) 241_{seven} (c) 500_{seven} (d) 4120_{seven}

    Then express each in decimal notation.

22. Below is the beginning of a chart of the numerals for the

numbers from 0 to 110 (base ten). From here on, in indicating base ten, write it in words ("base ten") rather than as a numeral. This is necessary since 10 means one of the base, which may be 7 or any number. Finish the chart, as you will need it later.

0	10	20	30	40	50	60	70	80	90	100	110
1	11	21	31	41	51	61	71				
2	12	22	32	42	52	62					
3	13	23	33								
4	14	24									
5	15										
6	16										
7	17										
8	18										
9	19										

23. Copy and complete the following table. Make a chart like the one in Ex. 22 to show the numerals from 0_{seven} to 110_{seven}.

0	10	20	30	40	50	60
1	11	21				
2	12			42		
3	13					
4	14					
5	15				55	
6	16					

Using Base Seven Numerals

24. When we learn to add numbers expressed in base ten we learn 100 "basic" combinations. These combinations are arranged in the chart below. Finish the chart.

Second addend

		0	1	2	3	4	5	6	7	8	9
	0	0	1	2	3	4	5	6	7	8	9
	1		2	3	4	5	6	7	8	9	10
	2			4	5	6	7	8	9	10	11
	3				6	7	8	9	10	11	12
First addend	4					8	9	10	11	12	13
	5						10	11	12	13	14
	6							12	13	14	15
	7								14	15	16
	8									16	17
	9										18

25. Compare the part of the table you have filled in with the part which is there now. What do you notice about the two parts?

26. Finish the chart below, showing the sums in base seven.

		Second addend						
		0	1	2	3	4	5	6
	0							
	1							
	2							
First addend	3							
	4							
	5							
	6							

27. When you add numbers expressed in base seven, how many basic combinations are there?

28. Draw lines in the addition chart for base seven like the lines in the chart for base ten. Are the two parts of the table alike?

29. Add these numbers in base ten:

$$\begin{array}{r} 25 \\ \underline{48} \end{array}$$

As you know, you "carry" when the sum of a column is ten or more.

$$\begin{array}{rl} 25 = & 2 \text{ tens} + 5 \text{ ones} \\ \underline{48} = & \underline{4 \text{ tens} + 8 \text{ ones}} \end{array}$$

$$6 \text{ tens} + 13 \text{ ones} = 7 \text{ tens} + 3 \text{ ones}$$

Add:

$$\begin{array}{rl} 24_{seven} = & 2 \text{ sevens} + 4 \text{ ones} \\ \underline{35}_{seven} = & \underline{3 \text{ sevens} + 5 \text{ ones}} \end{array}$$

$$5 \text{ sevens} + \text{nine ones} = 6 \text{ sevens} + 2 \text{ ones} = 62_{seven}$$

30. Is it possible to "carry" in addition in base seven just as you do in addition in base ten?

31. In base ten, you "carry" when the sum of a column is ? or more.

32. In base seven, you "carry" when the sum of a column is ? or more.

33. Add these numbers in base seven.

(a) $\begin{array}{r} 42_{seven} \\ \underline{13}_{seven} \end{array}$ (b) $\begin{array}{r} 65_{seven} \\ \underline{11}_{seven} \end{array}$ (c) $\begin{array}{r} 32_{seven} \\ \underline{25}_{seven} \end{array}$ (d) $\begin{array}{r} 254_{seven} \\ \underline{108}_{seven} \end{array}$ (e) $\begin{array}{r} 435_{seven} \\ \underline{625}_{seven} \end{array}$

34. For Ex. 33c, write out the addition to show how your answer was obtained. Express the numbers in Ex. 33 to base ten, and check your answers by adding in the usual way.

35. Subtract: $\begin{array}{r} 43_{seven} \\ \underline{15}_{seven} \end{array}$ Was your remainder 25_{seven}?

To subtract $\begin{array}{r} 43_{seven} \\ \underline{15}_{seven} \end{array}$, think "4 sevens + 3 ones = 3 sevens + ten ones

$$\begin{array}{rcl} & & \\ 1 \text{ seven} + 5 \text{ ones} & = & \underline{1 \text{ seven} \;\; + 5 \text{ ones}} \\ & & 2 \text{ sevens} + 5 \text{ ones} \end{array}$$

36. Subtract these numbers in base seven:

(a) $\begin{array}{r} 56_{seven} \\ \underline{14}_{seven} \end{array}$ (b) $\begin{array}{r} 61_{seven} \\ \underline{35}_{seven} \end{array}$ (c) $\begin{array}{r} 34_{seven} \\ \underline{25}_{seven} \end{array}$ (d) $\begin{array}{r} 456_{seven} \\ \underline{263}_{seven} \end{array}$

37. For Ex. 36c, show how the 34_{seven} is changed, in order to make it easier to subtract.

38. Complete the multiplication chart below for numbers in base seven. Are two parts of this table alike?

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0						
5							
6							

39. Multiply these base seven numbers.

(a) $\begin{array}{r} 52_{seven} \\ \underline{3} \end{array}$ (b) $\begin{array}{r} 34_{seven} \\ \underline{6} \end{array}$ (c) $\begin{array}{r} 421_{seven} \\ \underline{4} \end{array}$ (d) $\begin{array}{r} 21_{seven} \\ \underline{12}_{seven} \end{array}$

40. Here are some numbers in base seven. How would you write them in base ten?

(a) 43_{seven} (b) 526_{seven} (c) 304_{seven} (d) 260_{seven}

41. Change the numerals in the examples in Ex. 39 to numerals in base ten, and multiply again. Check your answers by changing the answers of Ex. 39 to base ten.

42. Explain the division of these base seven numbers. (The base is not written in the body of the work.)

$$\begin{array}{r} 454_{\text{seven}} \\ 6\overline{)4053}_{\text{seven}} \\ \underline{33}\phantom{53_{\text{seven}}} \\ 45\phantom{3_{\text{seven}}} \\ \underline{42}\phantom{3_{\text{seven}}} \\ 33\phantom{_{\text{seven}}} \\ \underline{33}\phantom{_{\text{seven}}} \end{array}$$

(a) $6|\overline{42_{\text{seven}}}$ (b) $6|\overline{420_{\text{seven}}}$

(c) $6|\overline{435_{\text{seven}}}$

43. Divide 501_{seven} by 2_{seven}.

44. Divide 652_{seven} by 5_{seven}.

TESTS FOR DIVISIBILITY

45. Look at the counting chart for numbers written in base 10. How can you tell which numbers are divisible by 2? How can you tell which are divisible by 5? By 10?

46. Now look at the counting chart for numbers written in base seven, and copy the first ten numbers which are exactly divisible by 2. Is there an easy way to tell whether a number is divisible by 2 from its expressions in base seven?

47. From the counting chart in base seven, copy the first five numbers which are divisible by seven. How can you tell whether a number written in base seven is exactly divisible by seven?

48. (a) In the counting chart for numbers written in base ten, look at the ones which are divisible by nine. What do you notice about the sum of the digits of each numeral? Can you guess the general rule?

    (b) Think of four numbers, larger than 100, which are divisible by 9. Add the digits in each numeral. Does the rule that you have just guessed work?

(c) Think of four numbers, larger than 100, which are not divisible by 9. See whether the rule you found works for those numbers.

(d) Can you tell how to decide whether a number is divisible by nine by adding the digits in its base ten numeral?

49. (a) In the counting chart for numbers written in base seven, look at the ones which are exactly divisible by six. Add the digits in each numeral. Do you notice a general rule?

(b) Think of four numbers larger than fifty, which are divisible by six. Write them in numerals in base seven, then add the digits in each numeral. Does your rule still work?

(c) Do the same for four numbers larger than fifty, which are not divisible by six. Does the rule work for these numbers?

(d) Can you tell how to decide whether a number is divisible by six by adding the digits in its base seven numeral?

50. Why should the test for divisibility by nine with numerals to base ten be like the test for divisibility by six with numerals to base seven?

51. (a) In the counting chart for base ten, choose five two-place numbers which are exactly divisible by 3. Find the sum of the digits for each number. Can you discover the general rule?

(b) Choose five numbers between ten and one hundred which are not exactly divisible by 3 and find the sum of the base ten digits. Does the rule still work?

(c) What seems to be a way to tell whether a number written in base ten is exactly divisible by 3?

(d) Look at the counting chart for base seven and choose five two-place numbers which are divisible by 3. Does the method you stated for numbers written in base ten seem to work for base seven numerals?

(e) Can you suggest another base for which this test works? If so, illustrate.

52. Write a number in base ten which can be exactly divided by 4. Write the same number in base seven. Can it be exactly divided by 4?

53. Think of a number which can be exactly divided by 5. Write the number in base seven numerals and in base ten numerals. Divide by 5, writing the quotient in base seven and also in base ten. Are the numerals the same? Do they represent the same number?

CHANGING FROM BASE TEN TO BASE SEVEN

You have learned how to change a number written in base seven numerals to base ten numerals. It is also easy to change from base ten to base seven. Let us see how that is done.

In base seven, the values of the places are 1, seven^1, seven^2, seven^3, and so on.

seven^1	7_{ten}
seven^2	49_{ten}
seven^3	343_{ten}

Example: Change 524_{ten} to base seven numerals.

Since 524 is larger than 343, first see how many $seven^3$ there are.

$$\begin{array}{r|l} & \overline{1} \\ 343 & 524 \\ & \underline{343} \\ & 181 \end{array}$$

The division shows there will be a 1 in the $seven^3$ position.

Now see how many 49's there are in 181.

$$\begin{array}{r|l} & \overline{3} \\ 49 & 181 \\ & \underline{147} \\ & 34 \end{array}$$

There will be a 3 in the $seven^2$ position.

Now see how many sevens there are in 34.

$$\begin{array}{r|l} & \overline{4} \\ 7 & 34 \\ & \underline{28} \\ & 6 \end{array}$$

There will be a 4 in the $seven^1$ position and 6 in ones' place.

So $524_{ten} = (1 \times 7^3) + (3 \times 7^2) + (4 \times 7^1) + (6 \times 1)$

$524_{ten} = 1346_{seven}$

54. Change 50_{ten} to base seven numerals.

55. Change 145_{ten} to base seven numerals.

56. Divide 1958_{ten} by ten. What is the quotient? What is the remainder? Divide the quotient by ten. What is the new quotient? Continue in the same way, dividing each quotient by 10, until you get a quotient of zero. How are the successive remainders related to the original number? Try the same process with $123,456,789_{ten}$. Try it with any other number.

57. Divide 524_{ten} by seven. What is the quotient? What is the remainder? Divide the quotient by seven, and continue as in Ex. 56, except that you divide by seven each time instead of ten. Compare the remainders with 524_{ten} written in base seven.

58. Can you give another method for changing from base ten to base seven?

OTHER BASES

You have studied numbers written in base ten and base seven. Now let us try some other bases.

59. How many symbols would there be in a system of notation in base five? base three? base four?

60. Draw sixteen x's on a paper. Draw lines around the groups you would use for base five. Then write the numeral showing the number of x's in base five. Be sure to write "five" after and below the numeral to show the base.

61. Now draw sixteen x's again, draw lines around groups of x's, and write the numeral in base four.

62. Draw sixteen x's again and show how to write the numeral in base three.

63. Make x's to show the numbers represented by

    (a) 13_{eight} (b) 23_{four} (c) 102_{three}

    (d) Do you have the same number of x's in all three groups of x's?

64. (a) How many threes are there in 20_{three}?

    (b) How many sixes are there in 20_{six}?

    (c) How many nines are there in 20_{nine}?

65. What is the smallest whole number which can be used as base for a system of number notation?

66. Here is part of a roll of tickets. Use different bases to record the number of tickets.

Tickets	Base ten	Base six	Base four	Base three
	1	1		
	2	2		
	3	3		
	4	4		
		5		
		10		

67. $467_{eight} = (4 \times 8^2) + (6 \times 8^1) + (7 \times 1)$

When we write the meaning of a numeral in this way, we say we are writing it in "expanded notation."

Write the following numbers in expanded notation.

(a) 638_{nine} (b) 245_{six} (c) 1002_{three}

68. How is each of these numbers written in base ten?

69. Write in expanded notation:

(a) 234_{five} (b) 103_{five} (c) 412_{five}

(d) If these numerals stand for amounts of money, what does each place represent?

Duodecimal Numerals

There are two bases of special interest. One of these is base twelve, which is the base of a system called the duodecimal system. We group many things by 12's, and call each group one dozen. We speak of a dozen eggs, a dozen rolls, a dozen pencils. When we have twelve twelves we call that one gross. Schools often buy pencils by the gross.

To write numbers in base twelve, you must have twelve symbols. You can make up symbols for ten and eleven, or use "t" and "e". The X's below are counted in base ten and in base twelve.

X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Base ten
1	2	3	4	5	6	7	8	9	t	e	10	11	12	13	14	Base twelve

70. Write in expanded notation

    (a) 146_{twelve} (b) $3t2_{\text{twelve}}$ (c) $47e_{\text{twelve}}$

    Then express these numbers in base ten.

71. Many people believe that twelve is a better base for a system than ten is. See if you can find out why they think so.

Binary Numerals

The other base which has special importance is base two, called the binary system. Just as we use base ten and the Babylonians used base sixty, some modern high speed computing machines use base two, or the binary system.

Suppose we use two as base. What symbols will there be? Only 0 and 1.

72. Draw three x's, draw lines around groups of two, and write the number of x's in base two.

73. Make a counting chart in base two, for the numbers from zero to seventeen. Remember the places will stand for powers of two.

74. Make an addition chart for base two. How many addition facts are there?

75. Make a multiplication chart for base two. How many multiplication facts are there? Would they be hard to learn?

76. Below are some numbers expressed in base two. The first is

written to show the meaning of the digit in each place. Write the others in that way.

$$1011_{\text{two}} = (1 \times \text{two}^3) + (0 \times \text{two}^2) + (1 \times \text{two}^1) + (1)$$

(a) 111_{two}

(b) 1000_{two}

(c) 10101_{two}

(d) 11000_{two}

77. What number does 2^3 represent? 2^2? 2^1? Write the powers of two, up to two^9, in base ten.

78. What numbers are represented in Ex. 76? Give your answer in base ten notation.

79. Add these numbers which are expressed in binary notation.

$$\text{(a)}\ \begin{array}{r} 101_{\text{two}} \\ \underline{10_{\text{two}}} \end{array} \quad \text{(b)}\ \begin{array}{r} 110_{\text{two}} \\ \underline{101_{\text{two}}} \end{array} \quad \text{(c)}\ \begin{array}{r} 10110_{\text{two}} \\ \underline{11011_{\text{two}}} \end{array} \quad \text{(d)}\ \begin{array}{r} 10111_{\text{two}} \\ \underline{11111_{\text{two}}} \end{array}$$

Check by expressing the numbers in base ten and adding in the usual way.

80. Subtract these base two numbers.

$$\text{(a)}\ \begin{array}{r} 111_{\text{two}} \\ \underline{101_{\text{two}}} \end{array} \quad \text{(b)}\ \begin{array}{r} 110_{\text{two}} \\ \underline{11_{\text{two}}} \end{array} \quad \text{(c)}\ \begin{array}{r} 1011_{\text{two}} \\ \underline{100_{\text{two}}} \end{array} \quad \text{(d)}\ \begin{array}{r} 11001_{\text{two}} \\ \underline{10110_{\text{two}}} \end{array}$$

81. Check by expressing the numbers in base ten and subtracting in the usual way.

82. When people operate some kinds of high speed computing machines they usually express numbers in base two. Change these base ten numbers to base two.

(a) 35 (b) 128 (c) 12 (d) 100

83. If you have a peg board and some match sticks, you can represent base two numbers on the board. Leave a hole blank for 0 and put in a match stick for one. Represent two numbers

on the board, one below the other, and try adding on the board.

84. Now you have worked with several different number bases. Do some have special advantages?

85. Have you ever seen a weighing scale which uses weights for balances? You put the thing to be weighed on one side and enough weights to balance it on the other side. Then you add up the weights you have used to find the weight of the thing.

Suppose you want a set of weights which will make it possible for you to weigh a package of 1 pound, 2 pounds, 3 pounds, and so on up to 15 pounds (no fractions). What is the fewest number of weights you will need, and what must their weights be?

86. Here is a set of cards you can use to do a trick.

1	
1	9
3	11
5	13
7	15

2	
2	10
3	11
6	14
7	15

4	
4	12
5	13
6	14
7	15

8	
8	12
9	13
10	14
11	15

Tell a person to choose a number between 1 and 15, and to pick out the cards containing that number and give them to you. By adding the numbers at the top of the cards he gives you, you can tell him the number he chose.

See if you can figure out how the trick works. Then see if you can make a set of cards which you can use for numbers from 1 to 31.

87. Here is a method of multiplication, different from the one you use.

25 x 34 = ?

	25	34	
even	12	68	cross out
even	6	136	cross out
	3	272	
	1	544	

Divide the numbers in the first column by 2, throwing away any remainders, and multiply the numbers in the second column by 2. Then cross out the numbers in the second column which are opposite even numbers in the first column, and add the numbers in the second column which are left. Does this give the correct product for 25 x 34? Try to figure out why this method works. (Remember base 2.)

88. Do you have an abacus in your classroom? If not, try to borrow one from a primary room or make one. Then make one to use for numbers expressed in base two.

UNIT III

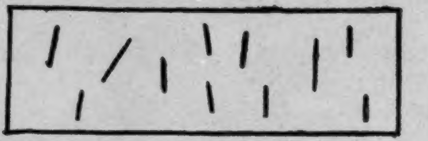
NATURAL NUMBERS AND ZERO

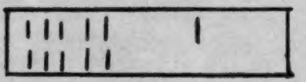
We Learned to Count

We use numbers in counting. We call them natural numbers. In Don's family there are five children. Jane buys twelve oranges at the store. There are five thousand four hundred eighty-five people in the auditorium. We have learned that numerals are names of numbers. If we are using our usual system of numeration, then the numerals 5, 12, 5485 are the names of the numbers mentioned. Man invented numerals when he recognized numbers in the world around him.

We know that there is the same number of people and noses in the room. If there are no wallflowers, there is the same number of boys and girls at a dance. There are 36 pupils in the room and the teacher has written the name of each pupil on a small card. The number of cards and the number of pupils is the same, assuming that the teacher has no blank cards. The number words, which we use form a standard set which we memorize in a certain order, and match with any set that we wish.

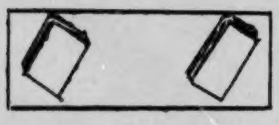
We may have learned about numbers in different ways. Some of us may have learned number names and then how to count. We learned that the last name we used gave us the number of the group counted. For example, here we have a collection or set of marks. We ask how many are there. By counting, we find that the last name we used in the matching process of a name with a mark was eleven. So we say the number of marks is eleven. If these are arranged in some way to form a pattern we may recognize a pattern of eleven without counting each mark. And that was [illegible] Is this easier? Do you need to count?

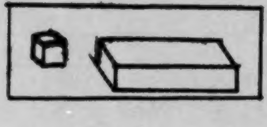




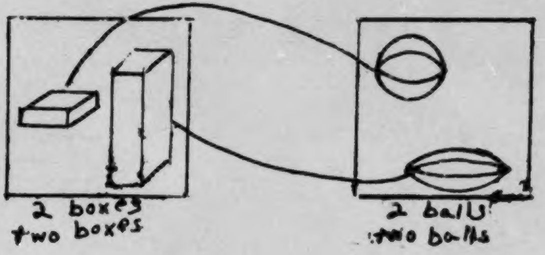
the way we learned to recognize groups without counting. We just knew how many were there. Many of us can recognize the number of a set of things without counting -- unless they are so mixed up that we can't see a familiar pattern.

Others of us may have learned the number of a set first without counting. We were told that here is a picture of two books and of two boxes.



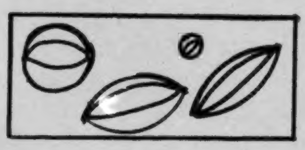


Perhaps we wondered what it was that was two but somehow we saw it. Things may have many properties in common -- like size, color, shape, use, taste. The same may be said of collections or sets of things. They have the property of containing the same number of members. This "sameness" is called number. Any time two sets of things can be matched like this --

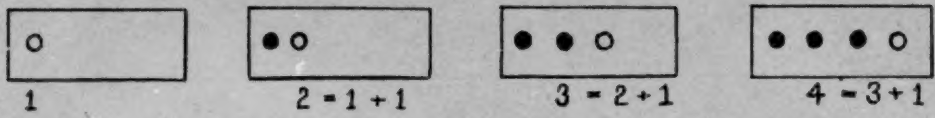


the two sets have a "same as" relationship. They both have the same number property. We use the same numeral to describe this number property.

We know that this is not a picture of 3 balls. We call the number of this set "four." We learned number names for different collections. We found that we could arrange these in an



order, each number being just one more than the one before it



starting with 1, 1 and 1 more is 2, 2 and 1 more is 3, 3 and 1 more is 4. It was then that we soon learned to group by tens, place value, and a system of numeration.

What would it have been like if we had learned a completely different name for each number?

We can, of course, count the number of units in two sets, provided that there is a term that applies. For instance, two cows and three horses make a set of five animals. Also three c's and two b's are five letters.

Exercises - 1

1. Which of the following are natural numbers:

   (a) 1 (b) 7 (c) 3% (d) .6 (e) 20 (f) 1/3

   (g) 3/3 (h) 100 (i) 8/4

2. Rearrange these natural numbers in their usual order.

   (a) 1, 2, 3, 6, 4, 5

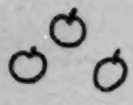
   (b) 1 + 1, 2 + 1, 1 + 3, 0 + 5, 1 + 6, 5 + 1, 5 + 3

   (c) IV, V, XI, VI, VII, VIII, X, IX

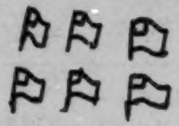
3. Which of the following numbers are not natural numbers between one and ten? 2, 5, 7, 8. between six and eleven? 4, 9, 11, 15? between one and fifty? 1, 15, 25, 28, 40. between one and ten? 13, 14, 22, 78, 86.

4. Here are sets of objects arranged so that the number of objects in the set may be recognised without counting. Write the name of the number of objects in each set.

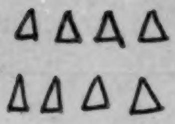
a)



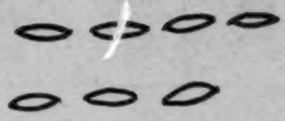
b)



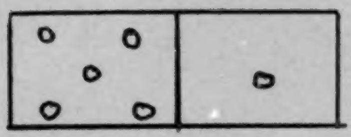
c)



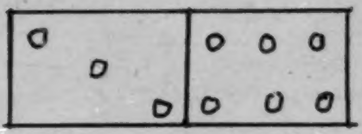
d)



e)

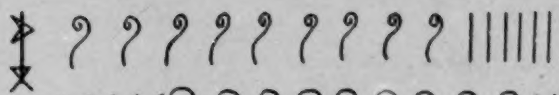


f)

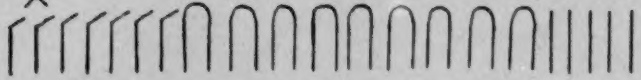


5. Rewrite the following Egyptian numerals, so that they may be easily read without counting

(a)



(b)

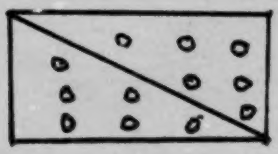


6. Show a grouping of marks which could easily be read without counting for the following

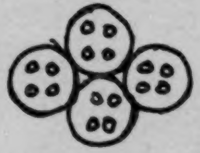
(a) 10 (b) 12 (c) 16 (d) 25

7. Tell how many dots are in the figures without counting each dot.

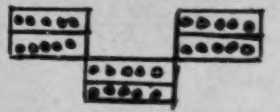
a)



b)



c)



8. Some people are standing in a room, in which there is a set of chairs. You want to find out if more chairs are needed. Is it necessary to count the people and the chairs to find out? What can you do to get quickly the needed information? Do you then know how many people are in the room?

9. A theater owner wants to know how many people attended the show last night. He knows the first ticket sold was numbered 60588 and the last ticket sold was 60735. Does he need to hire a man

to count the people as they come in? How many people attended the theater for that show?

10. A teacher has graded homework papers and recorded the grades in his gradebook. How can he quickly check to see if each student handed in his homework?

11. Tell how many more symbols are in one set than in the other. Do not count each set.

(a) ○ ○ ○ ○ ○ ○
× × × ×

(b) □ □ □ □ □ □ □
△ △ △ △

12. Make two sets of different figures and group them to show that one set is 5 more than the other.

13. When you count the number of people in the room, does it matter in what order you count? What must you be careful about in counting?

14. Sometimes we say that there are twice as many of one kind of thing as another. Give some examples in which you would know that there are twice as many of one kind as another without counting either set.

The Commutative Principle

If you have two boxes of pencils and in one of them there are 5 pencils and in the other there are two pencils, how many pencils do you have? What do you do to answer this question? If you say you "add" do you think, 5 + 2 or 2 + 5?

The arithmetic teacher read two large numbers to be added. John did not understand what his teacher said when she read the first number. He wrote the second number when she read it. Then he asked her to repeat the first number. When she read it again, John wrote it on his paper below the second number instead of above it. Will John find the same sum as the other students who heard all the dictation the first time? We say the sum of CCLXIV and CXXXII is the same as the sum of CXXXII and CCLXIV. In the binary system we find that the sum of 1101011 and 100111 is the same as the sum of 100111 and 1101011. Is this right?

Maybe you were told in elementary school that a good way to discover a mistake made in addition problems is to add again in "the other direction." If you "added down" then you might check your problem by "adding up." Even before that we learned that to put 3 balls with 2 balls gave us the same number of balls as putting 2 balls with 3 balls.

You have just recognized the *commutative principle* for *addition of natural numbers*. It means that the order in which we add two numbers doesn't make any difference in the sum of the two numbers.

$$3 \text{ added to } 4 \text{ is } 7 \text{ or } 4 + 3 = 7$$

$$4 \text{ added to } 3 \text{ is } 7 \text{ or } 3 + 4 = 7$$

Both are other names for 7. We can write

$$4 + 3 = 3 + 4.$$

The statement of this law in words is quite clumsy. It is simpler and clearer to say this in mathematical language:

"If a and b are natural numbers, then $a + b = b + a$."

We have just discussed the case in which $a = 4$ and $b = 3$.

Or, let's suppose Don's patrol goes on a trip. Twenty-nine boys go, each taking 3 dimes and 8 pennies for food. How many cents can be spent for food? How do you find the answer to this problem?

$$\text{This way} \quad \begin{array}{r} 29 \\ \times\ \underline{38} \end{array} \qquad\qquad \text{or this way} \quad \begin{array}{r} 38 \\ \times\ \underline{29} \end{array}$$

Bill's homeroom has a party. 38 boys and girls came to the party. They shared the cost of the party and each was asked to pay 29 cents. How much did the party cost? Do you find the answer

This way $\begin{array}{r} 38 \\ \times\ \underline{29} \end{array}$ or this way $\begin{array}{r} 29 \\ \times\ \underline{38} \end{array}$

Suppose we have 5 rows of chairs with 3 in each row. We decided to change the arrangement to make 3 rows of chairs with 5 in each row. Do we need more chairs?

x x x
x x x
x x x
x x x
x x x

(5 x 3)

x x x x x
x x x x x
x x x x x

(3 x 5)

The product of two natural numbers is the same, whether the first be multiplied by the second or the second be multiplied by the first. This statement is called the commutative principle for multiplication of natural numbers. It means that it makes no difference which number is the multiplier and which is the multiplicand. This statement may seem clumsy to you. Can you state this principle in symbols?

We can use this idea to detect mistakes we might make in multiplying one number by another. We found these products.

$$\begin{array}{r} 3927 \\ \underline{\times\ 485} \\ 15635 \\ 31416 \\ \underline{15708} \\ 1899595 \end{array} \qquad\qquad \begin{array}{r} 485 \\ \underline{\times\ 3927} \\ 3395 \\ 970 \\ 4365 \\ \underline{1455} \\ 1904595 \end{array}$$

As an application of the commutative property, we realize we have made a mistake. Find the mistake.

The idea of using letters to stand for any number whatsoever in stating general principles of arithmetic is a very useful part of mathematical language. There is a danger that the letter "x" and the multiplication sign may be mistaken for each other, so we frequently use a dot for multiplication. For example we can write "$4 \cdot 3$" for "4×3", "$a \cdot b$" for "$a \times b$."

In the Exercises the symbol "<" is read "less than". For example:

$$5 + 3 < 5 + 4$$

is read "5 + 3 is less than 5 + 4."

Exercises - 2

A

1. (a) Is 6 + 4 equal to 4 + 6?
   (b) Is 14 + 7 equal to 7 + 14?
   (c) Is 35 + 64 equal to 64 + 35?
   (d) Is 315 + 462 equal to 462 + 315?
   (e) Is 315 + 462 equal to 264 + 513?
   (f) Is 475 + 381 equal to 183 + 574?
   (g) Is $(13 + 32)_4$ equal to $(32 + 13)_4$?
   (h) Is $5 + 3 < 3 + 5$?
   (i) Is $5 + 2 < 3 + 5$?
   (j) Is $5 + 3 < 2 + 5$?
   (k) Is $5 + 4 < 3 + 5$?

B

1. (a) Is 72 + 31 equal to 31 + 72?
   (b) Is 16 + 52 equal to 25 + 61?
   (c) Is 58 + 94 equal to 94 + 58?
   (d) Is 465 + 332 equal to 564 + 233?
   (e) Is 735 + 254 equal to 537 + 452?
   (f) Is 851 + 367 equal to 158 + 763?
   (g) Is 58 + 94 commutative when written in the base seven?
   Insert a symbol which makes the following true statements.
   (h) 7 + 4   4 + 7
   (i) 9 + 6   7 + 9
   (j) 12 + 5   5 + 11
   (k) 46 + 81   64 + 18

(l) Is 5 + 3 < 4 + 5?

(m) Is 5 + 3 < 3 + 6?

(n) Is 6 + 3 < 4 + 5?

2. Add and use the commutative property to check addition

$\begin{array}{r} 465 \\ \underline{179} \end{array}$ $\quad$ $\begin{array}{r} 37,461 \\ \underline{73,135} \end{array}$

$\begin{array}{r} 73967 \\ \underline{81785} \end{array}$ $\quad$ $\begin{array}{r} (43)_5 \\ \underline{(32)_5} \end{array}$

3. (a) Is 6 x 5 equal to 5 x 6?

(b) Is 7 x 4 equal to 4 x 7?

(c) Is (3 + 2) · (5 + 2) equal to (5 + 2) · (3 + 2)?

(d) Is (3 + 1) · (2 + 3) equal to (4 + 1) · (2 + 2)?

(e) Is 24 x 43 equal to 43 x 24?

4. Multiply and use the commutative property to check multiplication

$\begin{array}{r} 36 \\ \times\ \underline{57} \end{array}$ $\quad$ $\begin{array}{r} 305 \\ \times\ \underline{84} \end{array}$ $\quad$ $\begin{array}{r} 476 \\ \underline{609} \end{array}$

$\begin{array}{r} 7604 \\ \times\ \underline{1008} \end{array}$ $\quad$ $\begin{array}{r} (34)_8 \\ \times\ \underline{(7)_8} \end{array}$

Which of the following are true

5. (a) 4 + 6 = 6 + 4

(b) 7 - 2 = 2 - 7

(c) 3 x 9 = 9 x 3

(d) 41 x 16 = 16 x 41

(l) 39 + 72 $\quad$ 39 + 72

(m) 35 + 53 $\quad$ 53 + 35

(n) 47 + 85 $\quad$ 81 + 51

2. Add and use the commutative property to check addition

$\begin{array}{r} 5,769 \\ \underline{1,785} \end{array}$ $\quad$ $\begin{array}{r} 465,781 \\ \underline{947,977} \end{array}$

$\begin{array}{r} (10111)_2 \\ \underline{(10101)_2} \end{array}$ $\quad$ $\begin{array}{r} (4t7e)_{12} \\ \underline{(124t)_{12}} \end{array}$

3. (a) Is 9 x 7 equal to 7 x 9?

(b) Is 43 x 56 equal to 56 x 43?

(c) Is (6 + 1) · (4 + 5) equal to (4 + 5) · (6 + 1)?

(d) Is (9 + 2) · (7 + 8) equal to (9 + 6) · (5 + 6)?

(e) Is (486) · (501) less than (501) · (486)?

4. Multiply and use the commutative property to check multiplication

$\begin{array}{r} 76 \\ \times\ \underline{98} \end{array}$ $\quad$ $\begin{array}{r} 47503 \\ \times\ \underline{20057} \end{array}$

$\begin{array}{r} (241)_5 \\ \times\ \underline{(32)_5} \end{array}$ $\quad$ $\begin{array}{r} (1011)_2 \\ \times\ \underline{(110)_2} \end{array}$ $\quad$ $\begin{array}{r} (465)_{12} \\ \times\ \underline{(372)_{12}} \end{array}$

Which of the following are true

5. (a) $(37 \times t)_{12} = (t \times 37)_{12}$

(b) (31 - 7) = (7 - 31)

(c) IV - I = V

(d) 49 x 63 = 64 x 48

(e) $8 \div 2 = 2 \div 8$

(f) $65 - 47 = 47 - 65$

(g) $72 \div 12 = 12 \div 72$

Insert a symbol to make the following true

6. (a) $25 + 17 \qquad 17 + 25$
   (b) $26 - 4 \qquad 4 - 26$
   (c) $60 \div 20 \qquad 20 \div 60$
   (d) $23 \times 12 \qquad 12 \times 32$
   (e) $7 \div 84 \qquad 84 \div 7$

(e) $78 + 54 = 53 + 79$

(f) $65 \div 5 = 5 \div 65$

(g) $72 \div 9 = 9 \div 72$

Insert a symbol to make the following true

6. (a) $47 + 182 \qquad 182 + 47$
   (b) $71 - 56 \qquad 56 - 71$
   (c) $625 \div 25 \qquad 25 \div 625$
   (d) $76 \times 67 \qquad 67 \times 76$
   (e) $25 \div 575 \qquad 575 \times 25$

Addition and subtraction are operations which can be performed on numbers. They are examples of binary operations, since they are performed on a pair of numbers. What other binary operations are frequently used in arithmetic?

The fact that the commutative principle is true for addition and multiplication is a property of these operations. Notice the similarity between the equations

$$a + b = b + a,$$

and

$$a \cdot b = b \cdot a.$$

We say that these operations are commutative. To find whether subtraction is commutative we must ask whether $a - b = b - a$ is true for any numbers a and b. Is it true for $a = 5$ and $b = 5$? Is division a commutative operation? Give an example illustrating your answer.

Exercises - 2a

7. Which of the following operations are commutative?

   (a) $1 + 2$
   (b) $6 + 8$
   (c) $7 \cdot 9$
   (d) $4 - 5$
   (e) $12 \div 3$
   (f) $9 - 4$

8. Which of the following activities are commutative?
   (a) To put on a hat and then a coat
   (b) To walk down the street and eat a hot dog
   (c) To pour red paint into blue paint
   (d) To pull up a chair and sit on it
   (e) To walk through a doorway and close the door.

9. What operations and activities can you list which are commutative? Which are not commutative? (You need not list more than five for each.)

10. Suppose we define a new operation, symbolized by $\vee$, like this:

$$a \vee b = a + b + (a \cdot b)$$

Compute $3 \vee 4$ and $4 \vee 3$. Is this operation commutative?

The Associative Principle

How do you add $\begin{array}{r}12\\ \underline{+3}\end{array}$? You think $2 + 3 = 5$, and bring down the 1. What have you actually done? Why does it work? You know that $12 = 10 + 2$. Your problem $\begin{array}{r}12\\ \underline{+3}\end{array}$, is $(10 + 2) + 3$. You actually found $10 + (2 + 3)$. We know $12 + 3 = 10 + (2 + 3)$. Let us try some other numbers.

What do we mean by $1 + 2 + 3$? Do we mean $(1 + 2) + 3$, or do we mean $1 + (2 + 3)$? Does it make any difference? We have said the order we add two numbers doesn't make any difference. Now we see that the way we group numbers to add them doesn't change the sum.

$$1 + (2 + 3) = 1 + 5 = 6$$

$$(1 + 2) + 3 = 3 + 3 = 6$$

We call this idea the associative principle of addition for natural numbers. Using both the commutative and associative properties of addition most anything goes in adding numbers. We can mix them up in just about any way we choose and we can still get the same answer. Maybe we don't stop to think about how we mix these up. For example,

> When asked to add 12 + 14 + 18, John thought, "12 + 14 = 26, and 26 + 18 = 44." This expression represents John's procedure — (12 + 14) + 18. Bill thought, "12 + 18 is 30, and 30 + 14 = 44." His thinking might be represented by (12 + 18) + 14. How did Bill know that (12 + 14) + 18 = (12 + 18) + 14?
>
> $(12 + 14) + 18 = 12 + (14 + 18)$ By the associative principle of addition
>
> $12 + (14 + 18) = 12 + (18 + 14)$ By the commutative principle of addition
>
> $12 + (18 + 14) = (12 + 18) + 14$ By the associative principle of addition

The commutative principle means we may change the order of any two numbers we are adding without changing the sum, that is $a + b = b + a$. The associative principle means that no matter how we may group numbers for purposes of adding pairs of numbers, the result is the same, that is, $(a + b) + c = a + (b + c)$.

When you multiply $2 \cdot (30)$ you actually compute $(2 \cdot 3) \cdot 10$. You know $30 = 3 \cdot 10$. Is $2(3 \cdot 10) = (2 \cdot 3) \cdot 10$?

The associative principle holds for multiplication with natural numbers. What do we mean by 2 x 5 x 4? Let's try some ways.

$(2 \times 5) \times 4 = 10 \times 4 = 40$

$2 \times (5 \times 4) = 2 \times 20 = 40$

Both give the same answer. We may group the factors of a product any way we please without changing the product. This is called the associative principle of multiplication for natural numbers, that is, $(a \times b) \times c = a \times (b \times c)$.

Since addition and multiplication are associative, there is no possibility of confusion when we write $1 + 2 + 3$ or $1 \cdot 2 \cdot 3$ omitting the parentheses. This is an example of mathematical slang which is allowed since it does not lead to any confusion.

However in $2 + 3 \cdot 4$ it does make a difference how we group the numbers since

$$(2 + 3) \cdot 4 = 5 \cdot 4 = 20$$

but

$$2 + (3 \cdot 4) = 2 + 12 = 14$$

Therefore it is wrong to omit the parentheses here and "$2 + 3 \cdot 4$" is nonsense unless we make some agreement about its meaning.

Exercises - 3

1. Show that the following are true.

Example: $(4 + 3) + 2 = 4 + (3 + 2)$

$$7 + 2 = 4 + 5$$

$$9 = 9$$

A	B
(a) $(4 + 7) + 2 = 4 + (7 + 2)$	(a) $(21 + 5) + 4 = 21 + (5 + 4)$
(b) $8 + (6 + 3) = (8 + 6) + 3$	(b) $(34 + 17) + 29 = 34 + (17 + 29)$
(c) $46 + (73 + 98) = (46 + 73) + 98$	(c) $436 + (476 + 1) = (436 + 476) + 1$
(d) $(6 \times 5) \times 9 = 6 \times (5 \times 9)$	(d) $(9 \times 7) \times 8 = 9 \times (7 \times 8)$
(e) $(24 \times 36) \times 20 = 24 \times (36 \times 20)$	(e) $(57 \times 80) \times 75 = 57 \times (80 \times 75)$

2. We now know that addition and multiplication of natural numbers

have the associative property. Let's look at subtraction. Is it associative? If it were for any natural numbers a, b, and c, $(a - b) - c = a - (b - c)$. Here is an example, try it.

Does $(10 - 7) - 2 = 10 - (7 - 2)$?

Does $(18 - 5) - 3 = 18 - (5 - 3)$?

Does subtraction have the associative property?

3. If division were associative, this would mean for any natural numbers, a, b, and c, $(a \div b) \div c = a \div (b \div c)$. Test this with this example: $(32 \div 8) \div 2 = 32 \div (8 \div 2)$. What conclusion do you come to? Make up another example to show again that you are right.

4. Rewrite these problems using the associative and commutative properties when necessary to make the addition easier. Use parentheses to show what additions are done first.

A

(a) $6 + 1 + 9$
(b) $5 + 7 + 2$
(c) $63 + 75 + 25$
(d) $26 + 72 + 4$
(e) $340 + 522 + 60$
(f) $45 + 15 + 63$
(g) $13 + 36 + 4$

B

(a) $72 + 90 + 10$
(b) $50 + 36 + 20$
(c) $28 + 75 + 25$
(d) $83 + 46 + 17$
(e) $3 + 5 + 7 + 15$
(f) $56 + 23 + 44 + 77$
(g) $18 + 16 + 24 + 2$

5. Rewrite these using the associative and commutative properties when necessary to make the multiplication easier. Use parentheses to show what multiplications are done first.

A

(a) 13 x 10 x 2
(b) 5 x 45 x 2
(c) 7 x 25 x 4
(d) 50 x 2 x 33

B

(a) 2 x 67 x 5
(b) 25 x 4 x 86
(c) 38 x 50 x 2
(d) 3 x 11 x 4

6. Is the following statement true? To multiply 2 by the product of 5 and 6 we can multiply 6 by the product of 2 and 5. Explain.
7. Is the following statement true? In order to double the product of 6 and 5, you double 6, double 5 and take the product of the doubles. Use parentheses to show what is being done. Explain the reasons for your answer.
8. Is the following statement true? In order to double the sum of 6 and 5, you double 6, double 5 and take the product of the doubles. Use parentheses to show what is being done. Explain the reasons for your answer.
9. How many ways can $2 + 3 \cdot 4 \div 2$ be interpreted by grouping the numbers in different ways?

The Distributive Principle

Eight girls and four boys -- twelve children altogether -- are planning a skating party. For a merrier party, each girl invites another girl and each boy invites another boy. The number of girls has been doubled. The number of boys has been doubled. Has the number of children been multiplied by two? by four? by twelve?

Altogether, there will be $(2 \cdot 8)$ girls and $(2 \cdot 4)$ boys or a total of $(2 \cdot 8) + (2 \cdot 4)$ children at the party. When the party was planned, there were $(8 + 4)$ children. The final number of children is the product of 2 and $(8 + 4)$. We see that --

$$(2 \cdot 8) + (2 \cdot 4) = 16 + 8 = 24$$

$$2 \cdot (8 + 4) = 2 \cdot 12 = 24$$

So, we can write

$$(2 \cdot 8) + (2 \cdot 4) = 2 \cdot (8 + 4)$$

You have been using this principle since the third grade. Take $\begin{array}{r}12\\ \underline{\times 3}\end{array}$

What do you actually think? You say to yourself, "$3 \cdot 2 = 6$, $3 \times 1 = 3$, and the product is 36." Your method is correct because

$$3(10 + 2) = 3 \cdot 10 + 3 \cdot 2$$

By the commutative principle it is also true that

$$(10 + 2)3 = 10 \cdot 3 + 2 \cdot 3$$

as we see when we change the order of multiplication.

This idea is familiar to us. We learned that $\begin{array}{r}375\\ \underline{\times \ \ 7}\end{array}$ is the same as $(7 \cdot 5) + (7 \cdot 70) + (7 \cdot 300)$. But instead of writing 375 as $(300 + 70 + 5)$ let's write 375 as $(145 + 230)$. Is $(7 \cdot 375)$ the same as $7 \cdot (145 + 230)$?

We can write $\begin{array}{r}83\\ \underline{\times 45}\end{array}$ as $(80 + 3) \cdot 45 = (80 \cdot 45) + (3 \cdot 45)$. This may be more familiar to some of us if we wrote $\begin{array}{r}45\\ \underline{\times \ 3}\end{array}$ and $\begin{array}{r}45\\ \underline{\times 80}\end{array}$.

In the more usual form we have

$$\begin{array}{r}45\\ \underline{\times\ 83}\\ 135\\ \underline{360\ \ }\\ 3735\end{array}$$

We recognize in the product that $45 \cdot 3 = 135$ and $45 \cdot 80 = 3600$.

This idea that we have been describing links together the operations of multiplication and addition. We refer to this idea as the distributive principle of multiplication over addition.

If we use letters to represent numbers, we can say

$$a(b + c) = a \cdot b + a \cdot c \quad \text{or} \quad (b + c) \cdot a = b \cdot a + c \cdot a = a \cdot b + a \cdot c.$$

Exercises - 4

1. Carry out instructions

A	B
(a) $(3 \cdot 9) + 6$	(a) $(4 \cdot 9) + 16$
(b) $3 \cdot (9 + 6)$	(b) $7(3 + 9)$
(c) $5 \cdot (2 + 7)$	(c) $(7 \cdot 3) + 9$
(d) $(5 \cdot 2) + 7$	(d) $(8 \cdot 6) + (3 \cdot 7)$
(e) $(8 \cdot 4) + (2 \cdot 3)$	(e) $(6 + 5) + (7 \cdot 9)$
(f) $(2 + 3) + (6 \cdot 8)$	(f) $(8 + 7) \cdot (9 \cdot 5)$
(g) $(3 + 9) \cdot (2 \cdot 3)$	(g) $(46 - 17) + 17$
(h) $(14 - 7) + 3$	(h) $(7 \cdot 3) + (7 \cdot 9)$
(i) $(7 \cdot 6) + (7 \cdot 9)$	(i) $16(7 + 6)$
(j) $7 \cdot (6 + 9)$	(j) $(16 \cdot 7) + (16 \cdot 6)$

2. Show that the following are true

Example: $3(4 + 3) = (3 \cdot 4) + (3 \cdot 3)$

$3 \cdot 7 \qquad 12 + 9$

$21 \qquad 21$

A	B
(a) $4(7 + 5) = (4 \cdot 7) + (4 \cdot 5)$	(a) $9(7 + 5) = (9 \cdot 7) + (5 \cdot 9)$
(b) $(3 \cdot 4) + (3 \cdot 8) = 3 \cdot (4 + 8)$	(b) $(11 \cdot 3) + (11 \cdot 4) = 11(3 + 4)$
(c) $(5 \cdot 2) + (5 \cdot 3) = 5(2 + 3)$	(c) $12(5 + 6) = (12 \cdot 5) + (6 \cdot 12)$
(d) $(6 \cdot 3) + (6 \cdot 2) = 6(3 + 2)$	(d) $(15 \cdot 6) + (15 \cdot 5) = 15(6 + 5)$
(e) $7(9 + 8) = (7 \cdot 9) + (7 \cdot 8)$	(e) $23(2 + 3) = (23 \cdot 2) + (23 \cdot 3)$
(f) $2(16 + 8) = (2 \cdot 16) + (2 \cdot 8)$	(f) $(3 \cdot 99) + (5 - 99) = 99(3 + 5)$
(g) $(3 \cdot 6) + (4 \cdot 6) = 6(3 + 4)$	(g) $128(10 + 20) = (128 \cdot 10) + (128 \cdot 20)$
(h) $9(1 + 2) = (9 \cdot 1) + (9 \cdot 2)$	(h) $(1000 \cdot 10) + (1000 + 20) = 1000(10 + 20)$

(i) $(10 \cdot 6) + (3 \cdot 10) = 10\ (6 + 3)$

(j) $(6 \cdot 8) + (6 \cdot 7) = 6(8 + 7)$

(i) $(8 \cdot 7) + (8 \cdot 7) = 8(7 + 7) = 7(8 + 8)$

(j) $15(3 + 3) = 3(15 + 15)$

3. Insert a symbol to make each a true sentence.

(a) $3(4 + 3) \quad (3 \cdot 4) + (3 \cdot 3)$

(b) $2(4 \quad 5) = (2 \cdot 4) + (2 \cdot 5)$

(c) $(5 \cdot 2) + (5 \quad 3) = 5(2 + 3)$

(d) $7(3 + 4) \quad (7 \cdot 3) + (7 \quad 3)$

(e) $(2 \cdot 7) + (3 \cdot \quad) = 7(2 + 3)$

(f) $(3 \cdot 5) + (3 \cdot 4) = 3(5 \quad)$

(a) $3(6 + 4) = (3 \cdot \quad) + (3 \cdot \quad)$

(b) $7(2 + \quad) = (7 \cdot \quad) + (\quad \cdot 3)$

(c) $(\quad \cdot 2) + (\quad \cdot 5) = 8(\qquad)$

(d) $(\quad \cdot 4) + (\quad \cdot 4) = \quad (6 + 7)$

(e) $11(\qquad) = (\quad \cdot 2) + (\quad \cdot 3)$

(f) $8(5 \quad 6) = (\qquad) + (\qquad)$

4. Using the distributive property rewrite each of the following:

Examples: (a) $5(2 + 3) = (5 \cdot 2) + (5 \cdot 3)$

(b) $(6 \cdot 4) + (6 \cdot 3) = 6(4 + 3)$

(a) $4(2 + 3)$

(b) $7(4 + 6)$

(c) $(9 \cdot 8) + (9 \cdot 2)$

(d) $6(13 + 27)$

(e) $(12 \cdot 5) + (12 \cdot 7)$

(a) $(5 \cdot 26) + (5 \cdot 7)$

(b) $8(14 + 17)$

(c) $7(213 + 787)$

(d) $(27 \cdot 13) + (27 \cdot 11)$

(e) $(18 \cdot 19) + (17 \cdot 18)$

5. Try these: Using the idea of the distributive property we can write

$36 + 42$ as $(6 \cdot 6) + (6 \cdot 7) = 6 \cdot (6 + 7)$

$18 + 15$ as $(6 \cdot 3) + (5 \cdot 3) = 3(6 + 5)$

Can you rewrite these in the same way?

Check your work to see that you are right.

Example: $18 + 15 = (6 \cdot 3) + (5 \cdot 3) = 3(6 + 5)$

$33 \qquad 18 + 15 \qquad 3 \times 11$

$33 \qquad 33$

A	B
(a) (6 + 4)	(a) (8 + 12)
(b) (12 + 9)	(b) (14 + 21)
(c) (10 + 15)	(c) (36 + 18)
(d) (24 + 18)	(d) (40 + 16)
(e) (28 + 32)	(e) (12 + 48)
(f) (21 + 14)	(f) (56 + 42)
(g) (25 + 15)	(g) (72 + 27)
(h) (3 + 6)	(h) (7 + 63)

6. Using the idea of the distributive principle, we can write 45 as (40 + 5) and 23 as (20 + 3). Then the product would be

$45 \cdot 23 = (40 + 5) \cdot (20 + 3) = 40(20 + 3) + 5(20 + 3) =$

$40 \cdot 20 + 40 \cdot 3 + 5 \cdot 20 + 5 \cdot 3 = 800 + 120 + 100 + 15 = 1070$

Check this result by multiplication of 45 and 23. Rewrite the following products in the same way, and check the results.

	A	B
(a)	78 · 45	86 · 34
(b)	13 · 76	53 · 19
(c)	567 · 84	623 · 72

7. Which of the following are true?

   (a) $3 + 4 \cdot 2 = (3 + 4) \cdot (3 + 2)$

   (b) $3 \cdot (4 - 2) = (3 \cdot 4) - (3 \cdot 2)$

   (c) $(4 + 6) \cdot 2 = (4 \cdot 2) + (6 \cdot 2)$

   (d) $(4 + 6) \div 2 = (4 \div 2) + (6 \div 2)$

8. Suppose we introduct the new symbol "$\wedge$" by the definition $a \wedge b = b^a$. For example, $2 \wedge 3 = 3^2 = 9$

   $3 \wedge 2 = 2^3 = 8.$

   Which of the following are true?

   (a) $2 \wedge (3 \cdot 4) = (2 \wedge 3) \cdot (2 \wedge 4)$

   (b) $(3 + 4) \wedge 2 = (3 \wedge 2) + (4 \wedge 2)$

   (c) $2 \wedge (12 \div 3) = (2 \wedge 12) \div (2 \wedge 3)$

The Closure Property

There is another property of natural numbers associated with the idea of addition and multiplication. If we add any two natural numbers (they may be the same number), our answer is also a natural number. For example,

$4 + 7 = 11$ and only 11
$5 + 5 = 10$ and only 10
$85 + 91 = 176$ and only 176

All are natural numbers.

We say that the set of natural numbers is closed with respect to addition, or closed under addition. That is, if we add any two natural numbers, we shall always get one and only one natural number as their sum.

The same thing is true for multiplication. The product of two natural numbers is one and only one natural number. For example, $2 \times 8 = 16$ and only 16. It isn't ever some other number like 38, 51, etc.

The operations of addition and multiplication have the property that when either is applied to a pair of natural numbers, in a given order, the result is a uniquely determined natural number.

We can say the set of natural numbers from one to ten is not closed to addition. (2 + 4) is a number less than 10 but is (6 + 5) less or equal to 10?

Is the set of natural numbers closed with respect to division? If we find the quotient of 8 and 2 or 8 ÷ 2, we get another natural number. But if we try 9 ÷ 2, then we do not get a natural number for an answer. We say that the set of natural numbers is not closed with respect to division. Too, we cannot always subtract one natural number from another and get a natural number. 16 - 4 = 12. All of these are natural numbers. But what about 4 - 15? Can our answer be a natural number? We say that this problem is impossible to solve if we must have a natural number as an answer.

Exercises - 5

1. Is the sum of two odd numbers always an odd number?
   Is the set of odd numbers closed under addition?
2. Is the set of even numbers closed under addition?
3. Is the set of all multiples of 5 (5, 10, 15, 20, etc.) closed with respect to addition?
4. What is true of the sets of numbers in Exercises 1, 2, and 3 under multiplication?
5. Are the following sets of numbers closed with respect to addition?
   (a) Set of natural numbers greater than 50?
   (b) Set of natural numbers from 100 through 999?
   (c) Set of natural numbers less than 48?
   (d) Set of natural numbers ending in 0?

6. Are the sets of numbers in Exercise 5 closed with respect to multiplication?
7. Are all sets of natural numbers which are closed with respect to addition also closed with respect to multiplication? Why?
8. Are any of the sets of numbers in Exercise 5 closed under subtraction?
9. Are any of the sets of numbers in Exercise 5 closed under division?

Inverse Operations

Often we do something and then we undo it. We open the door; we shut it. We turn on a light; we turn it off. We put on our coats; we take them off. We put two sets of things together into one set; we separate one set of things into two sets.

We call subtraction the inverse operation to addition. The inverse of adding 5 is subtracting 5.

What does the grocer do when you buy something for 31 cents, and pay for it with a dollar bill? Does he say, "100 - 31 = 69, here is 69 cents change."? No, he does not even mention the number 69. He counts out money into your hand, "32, 33, 34, 35, 40, 50, one dollar." He finds out how much to add to 31 in order to have 100. He answers the question:

$$31 + ? = 100$$

Can you subtract 23 from 58 by adding?

$$23 + ? = 58$$

You might think, " 3 + 5 = 8, 2 + 3 = 5, and the missing number is 35."

You have used addition to check subtraction. Can you use multiplication to check division?

We call division the inverse operation to multiplication. The inverse of multiplying by 5 is dividing by 5. The operation of multiplying 2 by 4 gives 8: $4 \times 2 = 8$. Now what can we do to 8 with 4 to give us 2? We divide. To describe the operation we can write $8/4 = 2$ because $4 \times 2 = 8$.

We have used this idea, too, as a check for division. How many 23's in 856? Study these operations.

$$\begin{array}{r} \underline{37} \\ 23)851 \\ \underline{690} \\ 161 \\ \underline{161} \end{array} \qquad\qquad \begin{array}{r} 23 \\ \times\ \underline{37} \\ 161 \\ \underline{690} \\ 851 \end{array}$$

When we ask the question, "What is 851 divided by 23?" we are seeking the answer to the question, "By what number must 23 be multiplied to obtain 851?" In the division and the check above, we see that 37 is the answer to both questions.

If a and b stand for two natural numbers, and a is greater than b there is a natural number, such that

$$a + x + b.$$

The number, x, is the number we find by subtracting b from a. We can explain the meaning of subtraction in terms of the equation $a + x = b$.

In a similar way, if a and b stand for natural numbers, then there may or may not be a natural number, x, such that

$$a \cdot x = b$$

If there is such a natural number, then x is the number we find by dividing b by a. We can explain the meaning of division in terms of the equation $a \cdot x = b$. If $b = 15$ and $a = 3$, then in dividing 15 by 3 we are seeking a number, x, such

$$3x = 15$$

Exercises - 6

1. Add the following numbers and check by the inverse operation:

A		B	
(a) 58 41	(b) 37 2400%	(a) 864 574	(b) 427 535
(c) 86 + 27		(c) Eight hundred and seventy-six plus four hundred and ninety-five is what?	
(d) One hundred and six plus eight hundred and ninety-seven is what?		(d) What is the sum of 32,098 and 80,605?	
(e) Find the sum of 798 and 508.		(e) Adding 20,009 to 89,991 gives what number?	

2. Subtract the following and check by addition:

(a) 86 - 2400%	(b) 67 - 28	(a) 916 - 805	(b) 1110 - 1010
(c) 167 - 78		(c) 899100% minus 6989	
(d) If one bookcase will hold 128 books and another 109 books, how many more books does the former hold?		(d) A theatre sold 4789 tickets one month and 6781 tickets the next month. How many more people came to the theatre the second month than came the first month?	
(e) If one building has 900 windows and another 811 windows, how many more windows does the first building contain?		(e) The population of a town was 19,891 people. Five years later the population was 39,110 people. What was the increase of population for the five years?	

3. Multiply the following numbers and check by the inverse operation:

(a) 2 · 241	(b) 734 · 9	(a) 213 · 23	(b) 518 · 76

(c) 20 · 841

(d) 239 · 37

(e) What is the product of 678 and 49?

(c) 509 · 4800%

(d) What is 87 times itself?

(e) If one truck will carry 2099 boxes, how many boxes will 79 trucks carry?

4. Do the indicated divisions and check by multiplication:

(a) $\frac{96}{3}$

(b) $\frac{936}{900\%}$

(c) $\frac{168}{12}$

(d) $\frac{357}{21}$

(e) At a party there were 312 pièces of candy. If there were 24 children at the party, how many pieces of candy could each child have?

(a) $\frac{70300\%}{13}$

(b) Divide 20972 by 107

(c) Take 214 into 108712.

(d) How many racks are needed to store 208 chairs, if each rack holds 16 chairs?

(e) A girl scout troop has 49 members. Each member is to sell boxes of cookies. If the troop has 588 boxes to sell, how many boxes will each girl have to sell in order to sell them all?

5. Find simpler name for:

(a) 39 - (2 + 5)

(b) (9 - 3) + 15

(c) 119 - (20 - 6)

(d) (20 + 11) - (6 - 2)

(e) (5 · 16) + 800%

(f) (35 · 42) - 8

(g) 9 + 201 + (128 · 239)

(h) $\frac{9120}{95} - 9600\%$

(a) 299 - (97 + 105 + 25)

(b) 973 + 728 - (728 - 27)

(c) (16 x 24) - 119

(d) (18 x 46) - (17 x 47)

(e) $\frac{3307949}{149} - 20100\%$

(f) $(\frac{625}{25}) \times (\frac{50}{25})$

(g) $(19 \cdot 17) + (\frac{5184}{7200\%}) - (35 + 37)$

(h) $\frac{(104 + 21)}{5(2 + 3)}$

(i) $\frac{8199}{900\%} + \frac{6488}{800\%}$

(j) $\frac{2460375}{135} + 1$

(i) $\frac{(610 + 14 + 205) - (4 \cdot 25)}{(3 \cdot 3) \cdot 300\%}$

(j) $35 \times (12 + 700\%)$

6. Perform the following operations:

(a) Add 16 and 17. From the sum subtract 12.

(b) Subtract 24 from 89. To this difference add 19.

(c) Multiply 27 by 34. Divide the product by 9 and then add 100.

(d) Find the sum of 9, 9 and 9. From it subtract 4, 6 times.

(e) Take 308, divide it by 28. Multiply the quotient by 5. Subtract 9 from the product.

(a) Find the difference between 47 and 38. Divide this difference by 3 and then add 17.

(b) Divide 272 by 16, multiply the quotient by 12 and subtract 100 from the product.

(c) Multiply 12 times 13 and add 39. Divide the sum by the product of 2 and 7.

(d) Add 26 and 42 and divide the sum by 17. To this add 117 and divide this sum by 11.

(e) Find the difference between 87 and 49. Multiply this difference by 10 and subtract 40. Divide this difference by 68 and then add 6.

Betweenness

Earlier we talked about ordering natural numbers. We now locate the sequence of natural numbers as dots.

1 2 3 4 5 6 7 8 9 10
. . . . (How far can we go?)

Now we are ready to ask some questions about numbers between numbers. How many numbers are between 1 and 7? What did you do to find your answer? Let's try these exercises.

Exercises - 7

1. Here is a row of dots running across the page. Beginning at the left end, taking steps one after another in succession, label each dot with a name for a number. Starting, it might look like this:

. . . .
one two three four five

Continue labeling, as far as the edge of the page. Writing the word names makes the picture cumbersome. If we again use the usual numeration, we will have

. . . .
1 2 3

This is called a number scale. Finish labeling the dots of the number scale.

(a) Write a number scale, using the binary numeration.

(b) Write a number scale using the seven system.

In the remaining questions use the decimal numeration:

(c) What numeral is the label for the dot nearest the right hand edge of the page?

(d) What numeral is the label for the dot just to the right of the dot labeled '7'?

(e) What numeral is the label for the dot just to the left of the dot labeled '7'?

(f) Between the dots labeled '6' and '8' there is only one dot, the dot labeled '7'. How many dots are between the dots labeled '11' and '13'?

(g) How many dots are between the dots labeled '2' and '4'?

(h) How many dots are between the dots labeled '5' and '8'?

We have more numerals which we have not used as labels. What numeral should be the label for the first dot beyond the paper's edge?

(i) What numeral should be the label for the second dot beyond the paper's edge?

(j) If the label '21' is given to the fifth dot beyond the paper's edge, where does the label '47' appear?

2. (a) How many dots between the dots labeled '1' and '12'?

   (b) How many dots between the dots labeled '12' and '1'?

   (c) How many dots between the dots labeled '51' and '58'?

   (d) How many dots between the dots labeled '27' and '3'?

   (e) How many dots between the dots labeled '84' and '532'?

   (f) How many dots between the dots labeled '38271' and '52964'?

3. At a drill all students line up in a single file and count off in one, two, three fashion. At her turn, Mary says, "eleven." At his turn, Tom says, "seventy-three." How many students are between Mary and Tom?

4. In a list of town voters, arranged alphabetically, Mrs. Beach is listed as number 197 and her sister, Mrs. Warren, is number 15841. How many names are on the register between the names of the sisters?

5. If a number scale is labeled from left to right, the dot labeled '8' will lie on the left of the dot labeled '10'. Since the number 8 is smaller than the number 10, we use the numeral '8' as a label before we use the numeral '10'. We have seen that symbols can be used to say that the number 8 is less than the number 10. We have written "8 < 10". The number scale shows us this sentence when it shows that the dot labeled '8' lies on the left of the dot labeled '10'.

   (a) The dot labeled '12' lies on the left of the dot labeled '17'. Does that mean that 12 < 17?

   (b) How do the dots labeled '482' and '516' lie?

(c) How many dots between the dots labeled '31' and '35'?

(d) What is the label on the dot midway between '31' and '35'?

6. In a stadium, the benches on the same level are labeled with numerals in a number scale. Don is sitting in the seat labeled '24', and Ed is sitting in the seat labeled '37'. Several very fat people want to sit between Don and Ed. Each one of the fat people needs two seats. How many fat people can squeeze in between Don and Ed?

7. There are uniform notches on a shelf. Each one holds a regular size box. An economy size box needs three spaces. Notch labeled '52' and notch labeled '142' are filled, but the shelf between is empty. How many regular size boxes may be put in? How many economy sized boxes?

8. Parking spaces in a factory parking lot are labeled like a number scale. Two cars are parked in the spaces labeled '57' and '80'. The spaces between are empty. Can a fleet of 12 trucks be parked between the two cars, if one truck occupies two spaces?

9. If a and b are numbers, and $a < b$, can it be true that $b < a$? Is it possible that $a = b$? If a, b, and c are numbers, and $a < b$ and $b < c$, then what is the relation between a and c? State your answer in the form:

   If $a < b$ and $b < c$, then ?

10. Which is larger, 3 or 7? If you add 2 to each number how do the results compare? Is there a general law? If $a < b$, then what is the relation between $a + c$ and $b + c$? State your answer in the form:

    If $a < b$, then ?

11. If a, b, and c are numbers, and b is between a and c, can c be between a and b? Is b between c and a? If b is between a and c, and a is between b and d, where d is a number, what is the relation among b, c, and d?

The Number One

The number 1 is the smallest of the natural numbers. It has several special properties which should be noticed. First, all the natural numbers may be built from 1 by addition; as we have seen: $1 + 1 = 2$, $1 + 2 = 3$, $1 + 3 = 4$, etc. Second the product of any natural number and 1 is that natural number: $1 + 1 = 1$, $1 \cdot 2 = 2$, $1 \cdot 3 = 3$, etc. It is therefore sometimes called the identity element for multiplication, since no number is changed when you multiply it by 1. Also if you divide by 1, the natural number is not altered.

As a matter of fact, if you assume that $1 \cdot 1 = 1$, you can use the distributive property to show that 1 times any natural number is itself. For example, suppose you wished to show that $1 \cdot 5 = 5$. Then you could write:

$$1 \cdot 5 = 1 \cdot (1 + 1 + 1 + 1 + 1) = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 =$$
$$1 + 1 + 1 + 1 + 1 = 5$$

Another property that the number 1 has is that $1^{50} = 1$, $1^{158} = 1$, in fact, any power of 1 is 1. Why does this follow from what we had stated above?

The Number Zero

The number zero is even more special than the number 1. It can be used to count in the sense that if you have no apples, you can express that fact by saying that you have zero apples. It is very useful in our notation for numbers since it serves often as a place holder: 11 is quite different from 101 and .0035 is not .35 .

What happens when we multiply, divide, add and subtract with zero? First $3 \cdot 0 = 0$ since this can be interpreted to be three zeroes and if three people have no apples each, they have amongst them no apples.

But 0 • 3 does not have a comparable meaning -- it makes no sense to speak of "no sets of three apples each." But we define 0 • 3 to be zero since we want multiplication to be commutative. Thus zero times any natural number is zero. Also it is natural to define $0 \cdot 0 = 0$. We might say that if we take the set of unicorns and the set of natural numbers between 1 and 2, and write none of them, we obtain a set of no elements.

It is also true that if the product of two whole numbers is zero, one or both of them must be zero. This holds since the product of two natural numbers is a natural number; that is, if neither of them were zero, their product could not be.

If we add zero to any natural number we get the number again. If you have no apples and I have three, we have three apples between us. The order in which we add them does not matter. We could express this by:

$$a + 0 = 0 + a = a,$$

for any natural number a. It is true even if a is zero. Hence a could be any whole number. Similarly $a - 0 = a$.

Can we divide by zero? We know that 6/2 is 3 because $3 \cdot 2 = 6$. So if 3/0 is a number it should be one which when multiplied by zero gives 3. All the numbers we have had so far give zero when multiplied by zero. It would be very strange to have a number which when we multiply it by zero we obtain 3. Another difficulty would be that if 3/0 were a number it would be equal to 1/0 since we could divide numerator and denominator by 3. Then 3/0 and 1/0 would be equal and yet when you multiplied the first by 0 you would get 3 and the other by zero you would get 1. For these and other reasons, we exclude division by zero.

Can we divide zero by any number? What about 0/3? This is the number which when multiplied by 3 gives zero. So 0/3 is zero, which is a number.

We have stated the properties of zero here in terms of natural numbers. But they all hold for other numbers as well.

Exercises - 8

1. If the product of two whole numbers is zero, one of them is zero. Is it true that if the product of two whole numbers is 1, one of them is 1? Is it true that if the product of two whole numbers is 2, one of them is 2? Will the answer be "yes" for any natural number in place of 2?
2. Answer the question at the close of the section on the number 1.
3. What is 3 - 3 equal to? What is n - n equal to?
4. If a + c = a, what must the number c be?
5. Should we consider 0/0 to be a number? Why or why not?

UNIT IV

FACTORING AND PRIMES

Do you know what the word "factor" means? The idea is familiar even if the word is not. We know that $5 \times 2 = 10$. Or we may write this as "$5 \cdot 2 = 10$." We call the 5 and 2 factors of 10; 6 and 7 are factors of 42, that is $6 \cdot 7 = 42$. Instead of calling one the multiplicand and one the multiplier, we can give them both the same name -- factor. Also, $42 = 2 \times 3 \times 7$ and we call 2, 3, and 7 the factors of 42. Really, does it make any difference which is which? From our understanding of the commutative property of multiplication of natural numbers we know that if the order is changed, the product is still the same. Whether it be 5×2 or 2×5, the product is still 10.

What are the factors of 96? Suppose we think one of the factors is 8. Then 96 divided by 8 is 12. We know that two factors of 96 are 8 and 12, for $8 \times 12 = 96$. Are there other factors of 96?

Exercises - 1

A

1. Factor the following:

   (a) 8 (b) 26 (c) 40

   (d) 54 (e) 56 (f) 72

   (g) 7

2. Using the principle that $n \cdot 1 = n$ if n is any number, find simpler names for these numbers.

   (a) $3(5 - 4)$

   (b) $(29 - 28) \cdot 5$

   (c) $\frac{2}{3} \cdot (56 - 55)$

B

1. Factor the following:

   (a) 16 (b) 28 (c) 144

   (d) 260 (e) 100 (f) 91

   (g) 13

2. Using the principle that $n \cdot 1 = n$ if n is any number, find simpler names for these numbers.

   (a) $8 \cdot \frac{3}{3}$

   (b) $\frac{5}{6} \cdot \frac{8}{8}$

   (c) $3 \cdot (86 - 85)$

(d) $57\%(75 - 74)$

(e) $7 \cdot \frac{6}{6}$

(f) $\frac{18}{18} \cdot 100\%$

(d) $(\frac{5}{5} \cdot 8) \times 9$

(e) $10 \cdot (\frac{4}{4} \cdot 3) \cdot (8 \cdot \frac{9}{9})$

(f) $\frac{50}{51} \cdot (76\frac{1}{2} - 75\frac{1}{2})$

3. For these use the natural numbers from 1 to 30.

   (a) Give the set of numbers that have the factor 1.

   (b) Give the set of numbers that have the factor 2.

   (c) Give the set of numbers that do not have the factor 2.

   (d) Give the set of numbers that may be factored in more than one way. Factor each number in this set as many ways as you can.

   (e) Give the set of numbers that can be factored only one way. Factor each number in this set. What can you say about the factors of a natural number that can only be factored one way?

We have talked about things having something in common. Let's write products of factors of numbers between 1 and 20. The even numbers between 1 and 20 are in Set I. The odd numbers between 1 and 20 are in Set II.

I	II
$1 \times 2 = 2$	$1 \times 3 = 3$
$1 \times 2 \times 2 = 4$	$1 \times 5 = 5$
$1 \times 3 \times 2 = 6$	$1 \times 7 = 7$
$1 \times 4 \times 2 = 8$	$1 \cdot 3 \cdot 3 = 9$
$1 \times 5 \times 2 = 10$	$1 \times 11 = 11$
$1 \times 6 \times 2 = 12$	$1 \times 13 = 13$
$1 \times 7 \times 2 = 14$	$1 \times 3 \times 5 = 15$
$1 \times 8 \times 2 = 16$	$1 \times 17 = 17$
$1 \times 9 \times 2 = 18$	$1 \times 19 = 19$

Notice that these even numbers all have the factor 2 in common, and none of these odd numbers has the factor 2. Do you suppose that all even numbers have the factor 2? Do you suppose that any odd number has the factor 2?

Exercises - 2

A	B
1. Pick out the even numbers.	1. Pick out the odd numbers.
37 64 31,766 56 101 420,681 102 2568 570,000	50 36,763 $(101)_3$ 738 829,700 $(210)_3$ 4683 10,110 500%
2. Tell whether these numbers are odd or even.	
(a) 2 x 5	(a) 3 x 2 x 6
(b) 3 + 7	(b) 5 + 7 + 1
(c) 6 x 5 x 3	(c) 4 x 7 x 13
(d) 2 + 16	(d) 128 + 36
(e) 7 + 8	(e) 266 + 627
(f) 5 x 13	(f) 3 x 3 x 7
(g) 257 + 361	(g) 25(7 + 9)
(h) 620 + 928	(h) (13 + 26)26
(i) 26 x 58 x 75	(i) (13 x 12) + 76
(j) 33 x 40 x 77	(j) 27 + (5 x 23)
(k) 5271 x 397 x 705	(k) 110 - 66
(l) 1729 + 5285	(l) 115 - 77
3. Perform the following operations:	
Make a counting chart for numbers from 1 through 10 in base two numeration. Circle the numerals for even numbers. How can you	Make a counting chart for numbers from 1 to 10 written in base two, three, four and five. In which system of numeration are even

recognise an even number written in base two numerals? An odd number?

numbers easily recognised? Do you have any hunches about even numbers written in systems with larger bases?

Prime Numbers

There are some numbers which have as factors only themselves and 1. For instance,

2 = 2 x 1

3 = 3 x 1

5 = 5 x 1

7 = 7 x 1

11 = 11 x 1

13 = 13 x 1

17 = 17 x 1

19 = 19 x 1

Any natural number which has only two factors -- itself and 1 -- is called a prime number. Although 1 has as factors only itself and 1, it is not considered a prime number. The numbers listed above -- 2, 3, 5, 7, 11, 13, 17, and 19, are all prime numbers. Numbers like 4, 6, 9, 12, 15, and 18, are not prime numbers. They are called composite numbers, for they have more than two different factors. For example, the factors of 4 are 1, 2, and 4. What are some other prime numbers? What are some other composite numbers?

Exercises - 3

1. List the numerals for all the prime numbers you can think of between 1 and 100.
2. Do you think you listed every one? You may have missed a few, or perhaps you have included the numerals of some numbers whose factors

you did not recognize. In about 200 B. C., there lived a man called Eratosthenes. This man invented a way to find prime numbers smaller than some number you have in mind. In this case the prime numbers are to be less than 100. To use Eratosthenes' method, we proceed as follows:

(a) Write in order the numerals for the odd natural numbers that are smaller than 100 beginning with 3.

(b) Starting with "3" cross out every third numeral. Do not cross out "3", but start counting with 4.

(c) Now, starting with "5" cross out every fifth numeral. Include the numerals already crossed out when you count. Some numerals will be crossed out more than once. Do not cross out the "5", but start counting with 6.

(d) Again, starting with "7", cross out every seventh numeral. Do not cross out the "7" but start counting with 8.

(e) The next numeral is "9". It is already crossed out. Skip it and go on to "11".

(f) Continue in this way until you have crossed out all possible numerals. The numerals left with the numeral "2" will be the names of the prime numbers less than 100.

Because numerals "drop out" this method is known as "the sieve of Eratosthenes."

Compare this list with your list in question one. Were you 100% correct? Keep this list in your notebook.

(g) Why is the numeral 2 added to the list of prime numbers? Could we have gotten the same list by writing the numerals for all the natural numbers? How would we begin, then? Do you understand our beginning is just a short cut?

(h) Was it necessary to continue to 31, where you would cross out every thirty-first numeral? At what point did you find that you were not crossing out any new numerals? When you moved on to a new step where the starting numeral had not been crossed out, but found that every other numeral in that step had already been crossed out, then you were finished.

3. (a) Using "the sieve of Eratosthenes", find the prime numbers less than 300.

(b) What numeral began the series in which the last numeral was crossed out? (See 2h above.)

(c) How many prime numbers are less than 300?

(d) How many prime numbers are between 1 and 100? between 100 and 200? between 200 and 300?

(e) Separate the natural numbers between 100 and 300 taken in order in groups of 50, in which group are the greatest number of primes? In which group of 50 do the number of primes increase over the previous group of 50?

(f) How many pairs of prime numbers are there such that their difference is 2? These are called prime twins.

More about Prime Numbers

Let's add some prime numbers --

$$\begin{array}{cccccc} 3 & 3 & 7 & 11 & 11 & 11 \\ \underline{5} & \underline{7} & \underline{5} & \underline{3} & \underline{5} & \underline{7} \\ 8 & 10 & 12 & 14 & 16 & 18 \end{array}$$

and we could keep on writing sums of prime numbers.

Are these sums always even numbers? Is this always true? Remember if you find one example which is not true, then the generalization cannot be made. Also, no matter how many examples we have, unless we have all possible examples, we do not have a proof.

Exercises - 4

1. Find these sums:

   (a) thirty-one plus nineteen

   (b) five plus twenty-nine

   (c) ninety-seven plus one hundred forty-nine

   (d) two hundred seventy-seven plus one hundred sixty-three

   (e) 199 + 233

   (f) 89 + 167

   (g) Do all of these examples ask for the sum of two prime numbers?

   (h) Is the sum an even number in each example?

2. Study these sums. Can you make a guess about them?

$4 = 2 + 2$
$6 = 3 + 3$
$8 = 5 + 3$
$10 = 5 + 5$

$26 = 23 + 3$
$28 = 17 + 11$
$30 = 17 + 13$
$32 = 19 + 13$

$100 = 47 + 53$
$102 = 41 + 61$
$104 = 43 + 61$
$106 = 23 + 83$

In 1742, a mathematician named Goldbach made a conjecture, or guess, about these even numbers, in fact, about all even numbers, except the number two, in a letter he wrote to a fellow mathematician, Euler. He guessed that every even number except 2, is the sum of two prime numbers.

   (a) Take a few examples and test Goldbach's conjecture. Take some numbers between 1 and 100, others between 100 and 200, others between 200 and 300.

   (b) Can you find one even number, other than 2 that is not the sum of two prime numbers?

   (c) Can you prove 'Goldbach's conjecture'? Try.

Another Property of Natural Numbers

In finding factors of numbers in Exercises 1, we gave pairs of numbers whose product was the given number. For example, $12 = 3 \times 4$, so two factors are 3 and 4. Another pair is 6 and 2. But the factors of 4 are 2 and 2. So we can say that the factors of 12 which are prime numbers are 3, 2, and 2, or $12 = 3 \times 2 \times 2$. Remember that we do not consider 1 a prime number. And the factors of 6 are 3 and 2. So again, $12 = 3 \times 2 \times 2$ and the factors are 3, 2, and 2. What kind of numbers are 3 and 2? They have what common property?

Let's study another example. If we can find factors of factors, we will do it. What are the factors of 24?

$24 = 12 \times 2$
$24 = (3 \times 4) \times 2$
$24 = (3 \times 2 \times 2) \times 2$
$24 = 3 \times 2 \times 2 \times 2$

$24 = 4 \times 6$
$24 = (2 \times 2) \times (2 \times 3)$
$24 = 2 \times 2 \times 2 \times 3$

$24 = 8 \times 3$
$24 = (4 \times 2) \times 3$
$24 = (2 \times 2 \times 2) \times 3$
$24 = 2 \times 2 \times 2 \times 3$

What do you observe? The factors of 24 are 2, 2, 2, and 3. One set is not in that order. Does that make any difference?

In finding factors of a number, we will find prime numbers. Every composite number can be factored into primes in only one way, except for order. This is called the *unique factorization property of natural numbers*.

Exercises - 5

1. Factor completely. (That is, find the prime factors.)

A

(a) 6
(b) 9
(c) 12
(d) 30
(e) 35

B

(a) 10
(b) 25
(c) 18
(d) 154
(e) 45

(f) 37

(g) 100

(h) 74

(i) 105

(j) 42

(k) 79

(l) 345

(m) 300

(n) 72

(o) 64

(f) 47

(g) 100

(h) 315

(i) 231

(j) 108

(k) 91

(l) 128

(m) 729

(n) 1000

(o) 5280

2. Examine the products in question 1. If any products use the same factor more than once, rewrite that product, taking advantage of the exponent notation.

3. Factor the numbers listed here in as many ways as possible using only two factors each time. Because of the commutative property, we shall say 3 x 5 is not different from 5 x 3.

A

(a) 6

(b) 8

(c) 24

(d) 100

(e) 150

B

(a) 10

(b) 16

(c) 72

(d) 81

(e) 216

4. Study 30 = 2 x 3 x 5. How should this set of factors be grouped to show that 30 = 2 x 15? to show that 30 = 6 x 5?

4. Why should a number such as 78 have more possible different pairs of factors than 77? or why should 210 have more possible different pairs of factors than 254?

5. (a) Factor 770 completely. Group the factors to show all the possible products that will equal 770.

(b) Factor each of the following numbers completely. Group the factors in each case to show all the possible ways the number is the product of two natural numbers.

(1) 42

(2) 66

(3) 78

(4) 12

(5) 18

(6) 48

(7) 49

(8) 75

(9) 64

5. (a) If a number when factored completely is made up of two different prime numbers, in how many different ways may the number be factored using different pairs of factors?

(b) If a number when factored completely is made up of three different prime numbers, in how many different ways may the number be factored using different pairs of factors?

(c) Fill in this chart.

Number	Complete Factorization	No. of Complete Factors	Different Ways to Factor Using Pairs	No. of Ways to Factor Using Pairs
55				
130	2x5x13	3	1x130,2x65 10x13,5x26	4
770				
2310				
28,014				

(c) Study

114 = 2 · 3 · 19

50 = 2 · 5 · 5

Why are there more possible ways to obtain 114 as a product of natural numbers than to obtain 50? Each of the numbers has 3 factors.

Is there any pattern in the number of ways such numbers may be factored using pairs of factors?

6. If I have 112 tulip bulbs to plant and would like to plant them to make a series of equal rows, what possible arrangements could I use?

6. If I have 1000 chairs to set up in an orderly fashion in a large auditorium, and want to make a series of equal rows, what possible arrangements could I make? If I would like the number of rows to be as close as possible to the number of chairs in each row, which possibility should I choose?

Greatest Common Factor (g.c.f.)

We have been looking for common properties in sets of things, that is, we have been finding something which each member of the set has. In our study of even numbers we saw that each even number has a factor of 2. So, we said even numbers have a common factor, 2. Let's find common factors for other sets of numbers. Is there a common factor for 10 and 15?

Factors of 10: 5 and 2

Factors of 15: 5 and 3

They have a common factor, 5.

Is there a common factor for 24 and 36?

Factors of 24: 2, 2, 2, 3

Factors of 36: 3, 3, 2, 2

Yes, they have common factors of 2, 2 and 3. We can say then that their largest factor in common, or their greatest common factor, is 2 x 2 x 3 or 12.

We can use factoring to help us change from one name for a fraction to another. For example, we know that

$$\frac{2}{4} = \frac{1}{2}$$

But let's use some of the things we have learned about greatest common factors. We write the factors of the numerator and denominator.

$$\frac{2}{4} = \frac{1 \cdot 2}{2 \cdot 2} = \frac{1}{2} \cdot \frac{2}{2}$$

But another name for $\frac{2}{2}$ is 1. So we can write

$$\frac{2}{4} = \frac{1}{2} \cdot 1$$

And we know that any number times 1 is itself. Then, $\frac{2}{4} = \frac{1}{2}$.

Study these:

(a) $\frac{18}{24} = \frac{6 \cdot 3}{6 \cdot 4} = \frac{6}{6} \cdot \frac{3}{4} = 1 \cdot \frac{3}{4}$ $\qquad \frac{18}{24} = \frac{3}{4}$ (g.c.f. is 6)

(b) $\frac{28}{36} = \frac{4 \cdot 7}{4 \cdot 9} = \frac{4}{4} \cdot \frac{7}{9} = 1 \cdot \frac{7}{9}$ $\qquad \frac{28}{36} \quad \frac{7}{9}$ (g.c.f. is 4)

(c) $\frac{36}{84} = \frac{3 \cdot 3 \cdot 2 \cdot 2}{7 \cdot 3 \cdot 2 \cdot 2} = \frac{3}{7} \cdot \frac{(3 \cdot 2 \cdot 2)}{(3 \cdot 2 \cdot 2)} = \frac{3}{7} \cdot 1$ $\qquad \frac{36}{84} = \frac{3}{7}$
(g.c.f. is 12)

Sometimes we find it difficult to recognize the greatest common factor for two or more numbers. We may use prime factors, just as we did in the last example above, to help us.

Exercises - 6

1. Factor each number completely and find the g.c.f.

(a) 35,21 and 49 (b) 21, 27 and 15 (c) 42, 147 and 105

(d) 60, 42 and 66 (e) 24, 60 and 84 (f) 78, 13 and 39

(g) 28, 56 and 14

2. Simplify, that is, carry out the indicated operations.

(a) $\frac{5 \cdot 3}{7 \cdot 3} =$ (b) $\frac{3 \cdot 5}{7 \cdot 3} =$ (c) $\frac{8 \cdot 2}{3 \cdot 8} =$

(d) $\frac{13 \cdot 4}{13 \cdot 5} =$ (e) $\frac{2 \cdot 5 \cdot 3}{11 \cdot 5 \cdot 3} =$ (f) $\frac{2 \cdot 5 \cdot 3}{2 \cdot 3 \cdot 13} =$

(g) $\frac{(2 \cdot 17) \cdot 11}{5 \cdot (2 \cdot 17)} =$ (h) $\frac{(5 \cdot 3) \cdot 6}{7 \cdot (3 \cdot 5)} =$ (i) $\frac{(2 \cdot 2 \cdot 5) \cdot 3}{(2 \cdot 2 \cdot 5) \cdot 7} =$

(j) $\frac{5 \cdot (3 \cdot 7 \cdot 3)}{(3 \cdot 3 \cdot 7) \cdot 11} =$ (k) $\frac{2 \cdot 15}{21 \cdot 7} =$ (l) $\frac{3 \cdot 5}{13 \cdot 2} =$

(m) $\frac{2 \times 2 \times 7 \times 3}{3 \times 2 \times 2 \times 2} =$

3. Find simpler names for these numbers.

(a) $\frac{10}{35}$ (b) $\frac{42}{60}$ (c) $\frac{49}{56}$ (d) $\frac{100}{166}$ (e) $\frac{36}{63}$ (f) $\frac{84}{112}$

(g) $\frac{100}{164}$ (h) $\frac{336}{633}$ (i) $\frac{5280}{5760}$ (j) $\frac{1456}{2184}$ (k) $\frac{945}{1080}$ (l) $\frac{8772}{20,468}$

Multiples of Numbers

In learning the multiplication facts, we learned multiples of numbers. For example, multiples of 4 between 1 and 40 are 4, 8, 12, 16, 20, 24, 28, 32, and 36. A multiple of 4 is a number that has 4 as a factor. Study the multiplication table again. What are multiples of other numbers? What numbers are multiples for different numbers? What numbers have the same multiple? Do you see any patterns in the multiples?

x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Exercises - 7

1. What multiples of 6 are less than 100?
2. What multiples of 14 are less than 100?
3. What multiples of 9 are between 250 and 300?
4. What multiples of 23 are between 300 and 350?
5. What natural numbers less than 10 have multiples whose decimal numerals end in 0, 2, 4, 6, 8?
6. What natural numbers less than 10 have multiples that are only even numbers?
7. What natural numbers less than 20 have multiples whose decimal numerals end in 0 or 5?
8. What natural numbers less than 20 have multiples whose numerals end in all the natural numbers less than 10?
9. What number does not have itself as a multiple?
10. What natural number less than 20 has multiples that are only odd numbers?
11. Can a natural number that is a composite number have a prime number as a multiple?
12. Write the names of six multiples of 12 using duodecimal numeration.
13. Write the names of six multiples of 6 using duodecimal numeration.
14. Write the names of six multiples of 2 using binary notation.

15. Outside white paint comes only in gallon cans. How many cans must be bought if 35 quarts are needed?

16. For refreshments at a campfire, each member is to receive 3 marshmallows. Marshmallows come in packages of 16, costing 13 cents a package. If 15 people are at the campfire, how many packages are needed?

17. If auditorium chairs come in sections containing 6 seats, how many sections will be needed for an audience of 100? of 150? of 200? of 201? of 202? of 203?

Least Common Multiple (l.c.m.)

We have learned that the greatest common factor for two or more numbers is the largest factor common to those numbers. The greatest common factor of 4, 6 and 8, is 2. It is the largest factor that is common to each.

We also know that:

Multiples of 4 are 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, etc.
Multiples of 6 are 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, etc.
Multiples of 8 are 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, etc.
What numbers are multiples of all three? Which is the smallest one? We call the smallest common multiple for two or more numbers their least common multiple.

We make use of this idea in finding like denominators in adding and subtracting fractions. It is true that we can use any common multiple. If we found the product of 4, 6, and 8, or 4 x 6 x 8, we would have another multiple, 192. However, it is easier to add fractions if we can find the smallest common multiple. Factoring helps us in finding it.

Find the least common multiple for 4, 6 and 8.

Factors of 4: 2, 2

Factors of 6: 2, 3

Factors of 8: 2, 2, 2

The least common multiple for these numbers must have all the different factors and any one of them as many times as it is a factor for any one of the numbers.

Least common multiple (l.c.m.) for 4, 6 and 8 is 2 x 2 x 2 x 3 or 24.

Study these examples:

(a) $\frac{2}{3} + \frac{5}{9}$ 27 is a common multiple of 3 and of 9.

$$\frac{2}{3} \cdot \frac{9}{9} + \frac{5}{9} \cdot \frac{3}{3} = \frac{18}{27} + \frac{15}{27} = \frac{33}{27} = 1 + \frac{6}{27} = 1 + \frac{2}{9}$$

By finding the least common multiple,

$$\frac{2}{3} \cdot \frac{3}{3} + \frac{5}{9} = \frac{6}{9} + \frac{5}{9} = \frac{11}{9} = 1 + \frac{2}{9}$$

(b) $\frac{3}{15} + \frac{5}{18} + \frac{7}{12}$

$$\frac{3}{15} \cdot \frac{12}{12} + \frac{5}{18} \cdot \frac{10}{10} + \frac{7}{12} \cdot \frac{15}{15} =$$

$$\frac{36}{180} + \frac{50}{180} + \frac{105}{180} = \frac{191}{180} = 1 + \frac{11}{180}$$

Finding least common multiple

15 = 5 x 3
18 = 3 x 3 x 2
12 = 2 x 2 x 3
l.c.m.: 5 x 3 x 3 x 2 x 2 or 180

(c)

$$\begin{array}{l} 7\frac{2}{9} = 7\frac{2}{9} \cdot \frac{2}{2} = 7\frac{4}{18} \\ 5\frac{1}{3} = 5\frac{1}{3} \cdot \frac{6}{6} = 5\frac{6}{18} \\ \underline{2\frac{1}{2} = 2\frac{1}{2} \cdot \frac{9}{9} = 2\frac{9}{18}} \\ \qquad\qquad 14\frac{19}{18} = 15 + \frac{1}{18} \text{ or } 15\frac{1}{18} \end{array}$$

Finding l.c.m.,
9 = 3 x 3
3 = 3 x 1
2 = 2 x 1
l.c.m.: 3 x 3 x 2 = 18

Exercises - 8

1. Find the multiples of the following numbers which are less than 100.

A	B
(a) 2, 3 and 4	(a) 2, 4 and 8

(b) 3, 6 and 9	(b) 6, 7 and 8
(c) 7, 8 and 9	(c) 11 and 5
(d) 13 and 3	(d) 14 and 12

2. Find the common multiples for the numbers listed in each part of Exercise 1.

3. Find the least common multiple for the numbers listed in each part of Exercise 1.

4. Find the least common multiple for:

A	B
(a) 6, 10 and 14	(a) 9, 15 and 21
(b) 20, 22 and 12	(b) 12, 14 and 16
(c) 70, 21 and 30	(c) 13, 15 and 17
(d) 5, 20 and 16	(d) 20, 40 and 50
(e) 9, 36 and 18	(e) 26, 12 and 39
(f) 5, 6 and 7	(f) 10, 75 and 45

5. (a) Find all the common multiples of 3, 4, and 8 which are less than 75.
   (b) Which is the least common multiple?
   (c) Which multiple is next greater than the l.c.m.?
   (d) Which multiple is next greater than the last one?
   (e) Do you have a hunch what the next two greater multiples will be?

   (a) What is the least common multiple of 5, 4 and 16?
   (b) What is another common multiple of 5, 4 and 16?
   (c) What common multiple is between 200 and 250?
   (d) What common multiple is between 550 and 600?
   (e) What hunch do you have about common multiples when compared with the least common multiple?
   (f) What is the greatest common multiple of 5, 4 and 16?

6. Perform the indicated operation.

(a) $\frac{2}{3} + \frac{7}{9}$

(b) $\frac{3}{14} + \frac{9}{16}$

(c) $\frac{7}{8} - \frac{3}{5}$

(d) $\frac{13}{16} - \frac{3}{20}$

(e) $\frac{9}{24} + \frac{9}{16}$

(f) $\frac{3}{4} + \frac{2}{5} + \frac{5}{6}$

(g) $\frac{5}{8} + \frac{7}{12} + \frac{1}{16}$

(h) $\frac{4}{28} - \frac{1}{7}$

(i) $\frac{1}{10} + \frac{15}{16} - \frac{3}{20}$

(j) $(\frac{3}{25} - \frac{1}{10}) - \frac{1}{15}$

(k) $\frac{3}{25} - (\frac{1}{10} - \frac{1}{15})$

(a) $\frac{5}{20} + \frac{3}{16}$

(b) $\frac{13}{24} - \frac{5}{18}$

(c) $\frac{2}{3} + \frac{3}{4} + \frac{4}{5}$

(d) $\frac{7}{8} + \frac{1}{14} - \frac{2}{35}$

(e) $(\frac{17}{18} - \frac{1}{9}) - \frac{2}{5}$

(f) $\frac{2}{3} - (\frac{4}{13} - \frac{11}{39})$

(g) $\frac{43}{50} - (\frac{2}{55} + \frac{19}{60})$

(h) $\frac{9}{14} - 6\%$

(i) $\frac{13}{22} + (\frac{37}{55} - \frac{10}{33})$

(j) $(\frac{33}{56} + \frac{33}{42}) - \frac{35}{96}$

(k) $(\frac{8}{15} + \frac{11}{18}) - (\frac{4}{9} + \frac{3}{10})$

UNIT IV-A

Supplementary Tests for Divisibility and Repeating Decimals

1. Introduction: This monograph is for the student who has studied a little about repeating decimals, numeration systems in different bases, and tests for divisibility (casting out the nines, for instance) and would like to carry his investigation a little further, under guidance. The purpose of this monograph is to give this guidance; it is not just to be read. You will get the most benefit from this material if you will first read only up to the first set of exercises and then without reading any further do the exercises. They are not just applications of what you have read, but to guide you in discovery of further important and interesting facts. Some of the exercises may suggest other questions to you. When this happens, see what you can do toward answering them on your own. Then, after you have done all that you can do with that set of exercises, go on to the next section. There you will find the answers to some of your questions, perhaps, and a little more information to guide you toward the next set of exercises.

The most interesting and useful phase of mathematics is the discovery of new things in the subject. Not only is this the most interesting part of it, but this is a way to train yourself to discover more and more important things as time goes on. When you learned to walk, you needed a helping hand, but you really had not learned until you could stand alone. Walking was not new to mankind -- lots of people had walked before -- but it was new to you. And whether or not you would eventually discover places in your walking which no man had ever seen before, was unimportant. It was a great thrill when you first found that you could walk, even though it looked like a stagger to other people. So, try learning to walk in mathematics. And be independent -- do not accept any more help than is necessary.

2. Casting out the nines. You may know a very simple and interesting way to tell whether a number is divisible by 9. It is based on the fact that a number is divisible by 9 if the sum of its digits is divisible by 9 and if the sum of its digits is divisible by 9, the number is divisible by 9. For instance, consider the number 156782. The sum of its digits is 1 + 5 + 6 + 7 + 8 + 2 which is 29. But 29 is not divisible by 9 and hence the number 156782 is not divisible by 9. If the second digit had been a 3 instead of 5, or if the last digit had been 0 instead of 2, the number would have been divisible by 9 since the sum of the digits would have been 27 which is divisible by 9. The test is a good one because it is easier to add the digits than to divide by 9. Actually we could have been lazy and instead of dividing 29 by 9, use the fact again, add 2 and 9 to get 11, add the 1 and 1 to get 2 and see that since 2 is not divisible by 9, then the original six digit number is not divisible by 9.

Why is this true? Merely dividing the given number by 9 would have tested the result but we would from that have no idea why it is would hold for any other number. We can show what is happening by writing out the number 156,782 according to what it means in the decimal notation:

$$1 \times 10^5 + 5\times10^4 + 6\times10^3 + 7\times10^2 + 8\times10 + 2 =$$

$$1\times(99999 + 1) + 5\times(9999 + 1) + 6\times(999 + 1) + 7\times(99 + 1) +$$

$$8\times(9 + 1) + 2.$$

Now by the distributive property, $5\times(9999 + 1) = 5\times9999 + 5\times1$ and similarly for the other expressions. Also we may rearrange the numbers in the sum since addition is commutative. So our number 156,782 may be written

$$1 \times (99999) + 5 \times (9999) + 6 \times (999) + 7 \times (99) + 8 \times 9 +$$

$$(1+5+6+7+8+2).$$

Now 99999, 9999, 999, 99, 9 are all divisible by 9, the products involving these numbers are divisible by 9 and the sum of these products is divisible by 9. Hence the original number will be divisible by 9 if (1+5+6+7+8+2) is divisible by 9. This sum is the sum of the digits of the given number. Writing it out this way shows that no matter what the given number is, the same principle holds.

Exercises

1. Choose four numbers and by the above method test whether or not they are divisible by 9. When they are not divisible by 9, compare the remainder when the sum of the digits is divided by 9 with the remainder when the number is divided by 9. Could you guess some general fact from this? If you can, test it with a few other examples.
2. Given two numbers. First, add them, divide by 9 and take the remainder. Second, find the sum of their remainders after each is divided by 9, divide the sum by 9 and take the remainder. The final remainders in the two cases are the same. For instance, let the numbers be 69 and 79. First, their sum is 148 and the remainder when 148 is divided by 9 is 4. Second, the remainder when 69 is divided by 9 is 6 and when 79 is divided by 9 is 7; the sum of 6 and 7 is 13, and if 13 is divided by 9, the remainder is 4. The result is 4 in both cases. Why are the two results the same no matter what numbers are used instead of 69 and 79? Would a similar result hold for a sum of three numbers?
(Hint: write 69 as 7 x 9 + 6)
3. If in the previous exercise we divided by 7 instead of 9, would the remainders by the two methods be the same? Why or why not?

4. Suppose in exercise 2 we considered the product of two numbers instead of their sum, would the corresponding result hold? That is, would the remainder when the product of 69 and 79 is divided by 9 be the same as when the product of their remainders is divided by 9? Why must this be true in general? Could they be divided by 23 instead of 9 to give a similar result? Could similar statements be made about products of more than two numbers?

5. Use the result of the previous exercise to show that 10^{20} has a remainder of 1 when divided by 9. What would its remainder be when it is divided by 3, by 99?

6. What is the remainder when 7^{20} is divided by 6?

7. You know that when a number is written in the decimal notation, it is divisible by 2 if its last digit is divisible by 2, and divisible by 5 if its last digit is 0 or 5. Can you devise a similar test for divisibility by 4, 8, or 25?

8. In the following statement, fill in both blanks with the same number so that the statement is true:

   A number written in the system to the base twelve is divisible by _____ if its last digit is divisible by _____. If there is more than one answer, give the others, too. If the base were seven instead of twelve, how could the blanks be filled in? (Hint: one answer for base twelve is 6)

9. One could have "decimal" equivalents of numbers in numeration systems to bases other than ten, though the word "decimal" would not be quite appropriate in this connection. For instance, in the numeration system to the base seven, $5(1/7) + 6(1/7)^2$ would be written .56 just as $5(1/10) + 6(1/10)^2$ would be written .56 in the decimal system. The number .142857142857... is equal to 1/7 in the

decimal system and hence in the system to the base seven would be written .1 . On the other hand, .1 = $(.04620462\ldots)_7$. What numbers would have terminating "decimals" in the numeration system to the base 7? What would the "decimal" equivalent of 1/5 be in the system to the base 7? (Hint: remember that if the only prime factors of a number are 2 and 5, the decimal equivalent of its reciprocal terminates)

10. Use the result of exercise 3 to find the remainder when 9 + 16 + 23 + 30 + 37 is divided by 7. Check your result by computing the sum and dividing by 7.
11. Use the results of the previous exercises to show that $10^{20} - 1$ is divisible by 9, $7^{108} - 1$ is divisible by 6.
12. Using the results of some of the previous exercises if you wish, shorten the method of showing that a number is divisible by 9 if the sum of its digits is divisible by 9.
13. See page 15.

3. Why does casting out the nines work? First let us review some of the important results shown in the exercises which you did above. In exercises 2, you showed that to get the remainder of the sum of two numbers, after division by 9, you can divide the sum of their remainders by 9 and find its remainder. Perhaps you did it this way (there is more than one way to do it; yours may have been better). You know in the first place that any natural number may be divided by 9 to get a quotient and remainder. For instance, if the number is 725, the quotient is 80 and the remainder is 5. Furthermore 725 = 80 x 9 + 5 and you could see from the way this is written that 5 is the remainder. Thus, using the numbers in the exercise, you would write 69 = 7x9 + 6 and 79 = 8x9 + 7. Then 69 + 79 = 7 x 9 + 6 + 8 x 9 + 7. Since the sum of two numbers is

commutative, you may reorder the terms and have $69 + 79 = 7 \times 9 + 8 \times 9 + 6 + 7$. Then, by the distributive property, $69 + 79 = (7+8) \times 9 + 6 + 7$. Now the remainder when $6 + 7$ is divided by 9 is 4 and $6 + 7$ can be written $1 \times 9 + 4$. Thus $69 + 79 = (7 + 8 + 1) \times 9 + 4$. So, from the form it is written in, we see that 4 is the remainder when the sum is divided by 9. It is also the remainder when the sum of the remainders, $6 + 7$, is divided by 9.

Writing it out in this fashion is more work than making the computations the short way but it does show what is going on and why similar results would hold if 69 and 79 were replaced by any other numbers, and, in fact, we could replace 9 by any other number as well. One way to do this is to use letters in place of the numbers. This has two advantages. In the first place it helps us be sure that we did not make use of the special properties of the numbers we had without meaning to do so. Secondly, we can, after doing it for letters, see that we may replace the letters by any numbers. So, in place of 69 we write the letter a, and in place of 79, the letter b. When we divide the number a by 9 we would have a quotient and a remainder. We can call the quotient the letter q and the remainder, the letter r. Then we would have

$$a = (q \times 9) + r$$

where r is zero or some natural number less than 9. We could do the same for the number b, but we should not let q be the quotient since it might be different from the quotient when a is divided by 9. We here could call the quotient q' and the remainder r'. Then we would have

$$b = (q' \times 9) + r'.$$

Then the sum of a and b will be

$$a + b = (q \times 9) + r + q' \times 9 + r'.$$

We can use the commutative property to have

$$a + b = (q \times 9) + (q' \times 9) + r + r'$$

and the distributive property to have

$$a + b = (q + q') \times 9 + r + r'.$$

Then if $r + r'$ were divided by 9, we would have a quotient which we might call q'' and a remainder r''. Then $r + r' = (q'' \times 9) + r''$ and

$$\begin{aligned} a + b &= (q + q') \times 9 + (q'' \times 9) + r'' \\ &= (q + q' + q'') \times 9 + r''. \end{aligned}$$

Now r'' is zero or less than 9 and hence it is not only the remainder when $r + r'$ is divided by 9 but also the remainder when $a + b$ is divided by 9. So as far as the remainder goes, it does not matter whether you add the numbers or add the remainders and divide by 9.

The solution of exercise 4 goes the same way as that for exercise 2 except that we multiply the numbers. Then we would have

$$\begin{aligned} 69 \times 79 &= (7 \times 9 + 6) \times (8 \times 9 + 7) \\ &= 7 \times 9 \times (8 \times 9 + 7) + 6 \times (8 \times 9 + 7) \\ &= 7 \times 9 \times 8 \times 9 + 7 \times 9 \times 7 + 6 \times 8 \times 9 + 6 \times 7 \end{aligned}$$

The first three products are divisible by 9 and by what we showed in exercise 2, the remainder when 69×79 is divided by 9 is the same as the remainder when $0 + 0 + 0 + 6 \times 7$ is divided by 9. So in finding the remainder when a product is divided by 9 it makes no difference whether we use the product or the product of the remainders.

If we were to write this out in letters as we did the sum, it would look like this:

$$\begin{aligned} a \times b &= (q \times 9 + r) \times (q' \times 9 + r') \\ &= q \times 9 \times q' \times 9 + q \times 9 \times r' + r \times q' \times 9 + r \times r' \end{aligned}$$

Again each of the first three products is divisible by 9 and hence the remainder when a x b is divided by 9 is the same as when r x r' is divided by 9.

We used the number 9 all the way above, but the same conclusions would follow just as easily for any number in place of 9, such as 7, 23, etc. We could have used a letter for 9 also but this seems like carrying it too far.

There is a shorter way of writing some of the things we had above. When letters are used, we usually omit the multiplication sign and write ab instead of a x b and 9q in place of 9 x q. Hence the last equation above could be abbreviated to

$$ab = qq'9 \times 9 + qr'9 + rq'9 + rr'$$

or

$$ab = 9 \times 9qq' + 9qr' + 9rq' + rr'.$$

But this is not especially important right now.

So let us summarize our results so far: The remainder when the sum of two numbers is divided by 9 (or any other number) is the same as the remainder when the sum of the remainders is divided by 9 (or the same other number). The same procedure holds for the product in place of the sum.

These facts may be used to give quite a short proof of the important result stated in exercise 13. Consider again the number 156,782. This is written in the usual form:

$$1 \times 10^5 + 5 \times 10^4 + 6 \times 10^3 + 7 \times 10^2 + 8 \times 10 + 2.$$

Now the result stated above for the product, the remainder when 10^2 is divided by 9 is the same as when the product of the remainders 1 x 1 is divided by 9, that is, the remainder is 1. Similarly 10^3 has a remainder 1 x 1 x 1 when divided by 9 and hence 1. So all the powers of ten have a remainder 1 when divided by 9. Thus, by the result stated above for the sum, the remainder when 156,782 is divided by 9 is the same as the remainder when 1 x 1 + 5 x 1 + 6 x 1 + 7 x 1 + 8 x 1 + 2 is divided by 9. This last is just the sum of the digits. Writing it this way it is easy to see that this works for any number.

Now we can use the result of exercise 13 to describe a check called "casting out the nines" which is not used much in these days of computing machines, but which is still interesting. Consider the product 867 x 935. We indicate the following calculations:

867	sum of digits:	21	sum of digits	3
935	sum of digits:	17	sum of digits	8

Product	810,645	Product: 3 x 8 = 24
Sum of digits	8+1+0+6+4+5 = 24	
Sum of digits	2+4 = 6	Sum of digits: 2 + 4 = 6

Since the two results 6 are the same, we have at least some check on the accuracy of the results.

Exercises

1. Try the method of checking for another product. Would it also work for a sum? If so try it also.
2. Explain why this should come out as it does.
3. If a computation checks this way, show that it still could be wrong. That is, in the example given above, what would be an incorrect product that would still check?
4. Given the number $5 \cdot 7^5 + 3 \cdot 7^4 + 2 \cdot 7^3 + 1 \cdot 7^2 + 4 \cdot 7 + 3$. What is its remainder when it is divided by 7? What is its remainder when it is divided by 6? by 3?
5. Can you find any short-cuts in the example above analogous to casting out the nines?
6. In a numeration system to the base 7 what would be the result corresponding to that in the decimal system which gives casting out the nines?

7. The following is a trick based on casting out the nines. Can you see how it works? You ask someone to pick a number -- it might be 1678. Then you ask him to form another number from the same digits in a different order -- he might take 6187. Then you ask him to subtract the smaller from the larger and give you the sum of all but one of the digits in the result. (He would have 4509 and might add the last three to give you 14). All of this would be done without your seeing any of the figuring. Then you would tell him that the other digit in the result is 4. Does the trick always work?

One method of shortening the computation for a test by casting out the nines, is to discard any partial sums which are 9 or a multiple of 9. For instance, in the example given, we did not need to add all the digits in 810,645. We could notice that $8 + 1 = 9$ and $4 + 5 = 9$ and hence the remainder when the sum of the digits is divided by 9 would be $0 + 6$, which is 6. Are there other places in the check where work could have been shortened? We thus, in a way, throw away the nines. It was from this that the name "casting out the nines" came.

By just the same principle, in a number system to the base 7 one would cast out the sixes, to the base 12 cast out the elevens, etc.

4. Divisibility by 11. There is a test for divisibility by 11 which is not quite so simple as that for divisibility by 9 but is quite easy to apply. In fact, there are two tests. We shall start you on one and let you discover the other for yourself. Suppose we wish to test the number 17945 for divisibility by 11. Then we can

write it as before

$$1\cdot10^4 + 7\cdot10^3 + 9\cdot10^2 + 4\cdot10^1 + 5.$$

The remainders when 10^2 and 10^4 are divided by 11 are 1. But the remainders when $10, 10^3$, 10^5 are divided by 11 are 10. Now 10 is equal to 11 - 1. $10^3 = 10^2(11 - 1)$, $10^5 = 10^4(11 - 1)$. That is enough. Perhaps we have told you too much already. It is your turn to carry the ball.

Exercises

1. Without considering 10 to be 11-1, can you from the above devise a test for divisibility by 11?
2. Noticing that 10 = 11 - 1 and so forth as above, can you devise another test for divisibility by 11?

We hope you were able to devise the two tests suggested in the previous exercises. For the first, we could group the digits and write the number 17945 as $1 \times 10^4 + 79 \times 10^2 + 45$. Hence the remainder when the number 17945 is divided by 11 should be the same as the remainder when 1 + 79 + 45 is divided by 11, that is 1 + 2 + 1 = 4. (2 is the remainder when 79 is divided by 11, etc.) This method would hold for any number.

The second method requires a little knowledge of negative numbers (either review them or, if you have not had them, omit this paragraph). We could consider -1 as the remainder when 10 is divided by 11. Then the original number would have the same remainder as the remainder when $1 + 7(-1)^3 + 9 + 4(-1) + 5$ is divided by 11, that is, when 5 - 4 + 9 - 7 + 1 is divided by 11. This last sum is equal to 4 which was what we got the other way. By this test we start at the right and alternately add and subtract digits. This is simpler than the other one.

Exercises

1. Test several numbers for divisibility by 11 using the two methods described above. Where the numbers are not divisible, find the remainders by the method given.
2. In a number system to the base 7, what number could we test for divisibility in the same way that we tested for 11 in the decimal system? Would both methods given above work for base 7 as well?
3. To test for divisibility by 11 we grouped the digits in pairs. What number or numbers could we test for divisibility by grouping the digits in triples? For example we might consider the number 157892. We could form the sum of 157 and 892. For what numbers would the remainders be the same?
4. Answer the questions raised in exercise 3 in a number system to base 7 as well as in a number system to base 12.
5. In the repeating decimal for 1/9 in the decimal system there is one digit in the repeating portion; in the repeating decimal for 1/11 in the decimal system, there are two digits in the repeating portion. Is there any connection between these facts and the tests for divisibility for 9 and 11? What would be the connection between repeating decimals and the questions raised in exercise 3 above?
6. Could one have a check in which 11's were "cast out"?
7. Can you find a trick for 11 similar to that in exercise 1 above?

5. Divisibility by 7. There is not a very good test for divisibility by 7 in the decimal system. (In a numeration system to what base would there be a good test?) But it is worth looking into since we can see the connection between tests for divisibility and the

repeating decimals. Consider the remainders when the powers of 10 are divided by 7. We put them in a little table:

n	1	2	3	4	5	6	7
Remainder when 10^n is divided by 7	3	2	6	4	5	1	3

If you compute the decimal equivalent for 1/7 you will see that the remainders are exactly the numbers in the second line of the table in the order given. Why is this so? This means that if we wanted to find the remainder when 7984532 is divided by 7 we would write

$$7 \times 10^6 + 9 \times 10^5 + 8 \times 10^4 + 4 \times 10^3 + 5 \times 10^2$$
$$+ 3 \times 10 + 2$$

and replace the various powers of 10 by their remainders in the table to get

$$7 \times 1 + 9 \times 5 + 8 \times 4 + 4 \times 6 + 5 \times 2 + 3 \times 3 + 2$$

We would have to compute this, divide by 7 and find the remainder. That would be as much work as dividing by 7 in the first place. So this is not a practical test but it does show the relationship between the repeating decimal and the test.

Notice that the sixth power of 10 has a remainder of 1 when it is divided by 7. If instead of 7 some other number is taken which has neither 2 nor 5 as a factor, 1 will be the remainder when some power of 10 is divided by that number. For instance, there is some power of 10 which has the remainder of 1 when it is divided by 23. This is very closely connected with the fact that the remainders must from a certain point on, repeat. Another way of expressing this result is that one can form a number completely of 9's, like 99999999, which is divisible by 23.

Exercise

Complete the following table. In doing this notice that it is not necessary to divide 10^{10} by 17 to get the remainder when it is divided by 17. We can compute each entry from the one above, like this: 10 is the remainder when 10 is divided by 17; this is the first entry. Then divide 10^2, that is, 100 by 17 and see that the remainder is 15. But we do not need to divide 1000 by 17. We merely notice that 1000 is 100 x 10 and hence the remainder when 1000 is divided by 17 is the same as the remainder when 15 x 10, or 150 is divided by 17. This remainder is 14. To find the remainder when 10^4 is divided by 17, notice that 10^4 is equal to 10^3 x 10 and hence the remainder when divided by 17 is the same as when 14 x 10 is divided by 17, that is 4. The table then gives the remainders when the powers of 10 are divided by various numbers:

	3	7	9	11	13	17	19	21	37	101	41
1	1	1	1			1					
10^1	1	3	1			10					
10^2	1	2	1			15					
10^3	1	6	1			14					
10^4	1	4	1			4					
10^5	1	5	1			6					
10^6	1	1	1			9					
10^7	1		1			5					
10^8	1		1			16					
10^9	1		1			7					
10^{10}	1		1			2					
10^{11}	1		1			3					
10^{12}	1		1			13					
10^{13}	1		1			11					
10^{14}	1		1			8					
10^{15}	1		1			12					
10^{16}	1		1			1					

Find what relationships you can between the number of digits in the repeating decimals for 1/3, 1/7, 1/9, 1/11, 1/13, etc. and the pattern of the remainders. Why does the table show that there will be five digits in the repeating portion of the decimal for 1/41? Will there be some other fraction 1/? which will have a repeating decimal with five digits in the repeating portion? How would you find a fraction 1/? which would have six digits in the repeating portion?

If you wish to explore these things further and find that you need help, you might begin to read some book on the theory of numbers. Also there is quite a little material on tests for divisibility in "Mathematical Excursions" by Miss Helen Abbott Merrill, Dover (1958).

continued from page 5.

Ex. 13. Show why the remainder when the sum of the digits of a number is divided by 9 is the same as the remainder when the number is divided by 9.

UNIT V

THE NON-NEGATIVE RATIONAL NUMBERS

1. Whole numbers and divisibility. You are familiar with the natural numbers 1, 2, 3, 4, 5, 6, and so on, and the number zero. These we have agreed to call the whole numbers. Later on we shall have what we call "negative numbers" as well and find that another name for the whole numbers is: "the non-negative integers", or "the positive integers and zero". But in this unit we shall just use the words "whole numbers" to describe the set 0, 1, 2, 3, etc.

Division is the "inverse" of multiplication; that is, 6/2 = 3 and 6 = 2 x 3 are two ways of expressing the same relationship. Also 6/4 = 1½ because 6 = 4 x 1½. When we divide 6 by 2, we obtain a natural number and we say "6 is divisible by 2". But when we divide 6 by 4 we do not obtain a natural number and we say "6 is not divisible by 4". The term "divisible" does not mean merely "you can divide" (this can usually be done - certainly in both cases above) but it means that both the divisor and quotient are natural numbers. Two other ways of saying "40 is divisible by 5" are "40 is a multiple of 5" and "5 is a factor of 40".

2. The fractional notation. The symbol 6/2 could have two meanings. It might be six halves or half of six, that is 6 x ½ or ½ x 6. The fact that these two are equal is called

the commutative property of multiplication. We see that six halves are 3, which is half of six.

$$\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = 1 + 1 + 1 = 3.$$

So we use the two meanings interchangeably.

1/4 is the number such that if you multiply it by 4, you obtain 1; that is 4x(1/4)=1. In a similar way, 20x(1/20) or (1/20)x20=1.

3/4 would be 3x(1/4) or (1/4)x3. We can also say that 3/4 is the number such that if you multiply it by 4, you obtain 3.

The quotient of any two natural numbers we call a rational number. Some examples are: 3/4, 7/2, 6/1, 125/789, 670000/3.

Exercises A

1. Give examples of the following kinds of numbers:
   - (a) natural numbers
   - (b) whole numbers
   - (c) non-negative numbers
   - (d) rational numbers
2. Express the relationships as products:
   - (a) (8/2)=4
   - (b) (21/7)=3
   - (c) (150/15)=10
   - (d) (29/5)=5 4/5
3. By what natural numbers is 144 divisible? What numbers are factors of 144?
4. List 10 multiples of 7.
5. List 5 multiples of 13.
6. What are two of the meanings of 3/8?

7. 2/3 is the number such that if it is multiplied by 3, 2 is obtained. Use this language to describe the following numbers:

   (a) 4/5   (b) 7/3   (c) 1/8   (d) 6/11   (e) 100/9

8. From the product: 2x3x5x17=510 we can say
   - (a) What numbers are factors of 510?
   - (b) 510 is divisible by what numbers?
   - (c) 510 is a multiple of what numbers?

9. See Exercise 8. By which of the following numbers is 510 divisible?

   4, 6, 8, 10, 11, 15, 20, 34, 51, 52

10. Assume a,b,c, and d are natural numbers. If axbxc=d, make as many statements as you can about factors and multiples involving 2 or more of the numbers a,b,c, and d. Is d a multiple of axb? Is bxc a factor of d?

3. Multiplication of rational numbers. In order to use rational numbers we must be able to multiply and add them, and the properties of multiplication and addition should be the same as far as possible for the rational numbers as for the whole numbers. Since multiplication is a little easier than addition, we shall consider it first. What should be the value of (1/3)(1/4)? It is one-third of one-fourth. In other words we would divide something into four equal parts and then each of these parts into three equal parts. We would

have in all 12 equal parts. Hence we should define the product $(1/3) \times (1/4)$ to be $1/12$. Similarly, $(1/6) \times (1/5) = 1/30$. This would suggest what the product should be for any natural numbers in place of 3 and 4. One way to express this would be to replace 3 and 4 by letters instead of other numbers and have it understood that the letters stand for any numbers. Then we would have

$$(1/a)(1/b) = 1/(ab),$$

where ab means the product of a and b.

Suppose we have two rational numbers whose numerators are not 1, such as $(3/4) \times (5/7)$. Then this could be written

$(3/4) \times (5/7) = 3 \times (1/4) \times 5 \times (1/7)$ by the definition of a rational number and the associative property.

$= 3 \times 5 \times (1/4) \times (1/7)$, by the commutative principle.

$= 15 \times (1/28) = 15/28$, using the value of the product of two rational numbers with one in each numerator.

This would work equally well with any natural numbers in place of 3, 4, 5, and 7. Expressed in letters it would be

$$(a/b)(c/d) = (ac)/(bd).$$

In words what does this mean?

Exercises B

1. Explain what is meant by each of the following: 7/12, 5/3, 10/6, 14/24.

2. Find the values of each of the following: (5/3) x 6, (7/4) x 4, (3b/b) x b for several whole numbers in place of b.

3. We know that 6/6 = 1, 20/20 = 1. Using this and assuming that the product of any rational number and 1 is the rational number, find the value of:

   (3/5) x (6/6), (7/10) x (20/20), (11/6) x (7/7).

4. Compute the products indicated in the previous exercise, using the definition of the product of two rational numbers.

5. Can the natural number 6 be thought of as the rational number 6/1? Why?

6. Find the products, using the form above for the product of (3/4)x(5/7), and giving the reasons for each step:

   (a) (1/2)x(3/5) (b) (2/3)x(3/4)

   (c) (5/6)x(8/9) (d) (1/4)x(2/3)x(7/8)

7. Using the definition of the product of two non-negative rational numbers, is the set of non-negative rationals closed with respect to multiplication?

8. Find the following products:

(a) 6x(3/11) (b) (2/9)x4 (c) (1/3)x(1/4)x(1/5)

(d) (2/3)x(7/8) (e) (7/8)x(5/7) (f) (1/4)x(8)x(3/6)x(9/4)

9. Suppose two equal rational numbers have equal denominators. What can you say about their numerators? Suppose the equal rational numbers had equal numerators, what could you say about their denominators?

10. State in words the method of finding the product of two rational numbers.

4. Equality of rational numbers. We know that 6/2 and 12/4 are two ways of representing the number 3. Are there different ways of representing any rational number? We know that the answer to this is "yes" since, for example 1/2=2/4. Here it is helpful to make a distinction that we made for natural numbers: there is a difference between a natural number and the symbol used to represent it. We call the symbol, the "numeral". Here when we want to make a distinction, we call the symbol the "fraction". If we were going to be very particular we would have written in the last section: a fraction which represents the product of two rational numbers is one whose numerator is the product of the numerators and whose denominator is the product of the denominators of the fractions which represent the given numbers. This, of course, is being altogether too particular. But it is useful

at times to have the word "fraction" for the symbol. For instance, we could say that the two fractions 1/2 and 2/4 represent the same rational number and so we call them equal. Also we should probably speak of the numerator and denominator of a fraction but not of a rational number. But this is awkward, too, and there is not likely to be any confusion when we speak of the numerator of a rational number, if we realize that it may have several numerators and that we are merely referring to the way it is written at the time.

We saw in exercises 3 and 4 above that (3/5)x(6/6) is on the one hand equal to (3/5)x1 which should be 3/5. On the other hand, if we multiply the numbers, we obtain 18/30. So 18/30 should be equal to 3/5. We could have used any natural number in place of 6 and seen, for instance, that 3/5=21/35. In fact, no matter what natural number k is, it would be true that 3/5=(3xk)/(5xk). We can write this more briefly as

$$(3/5)=(3k/5k)$$

We can multiply the numerator and denominator of any fraction by a natural number without changing the value of the rational number which it represents. Also, working from the right to the left, we can divide both numerator and denominator of any fraction by the same natural number without changing the rational number which it represents.

How do we find out whether two fractions represent the same rational number? Suppose we had 6/15 and 4/10, in which the numerator of one is not a divisor of the numerator of the other. One method would be to reduce each fraction to lowest terms, that is, in each fraction divide the numerator and denominator by any common factor. Then $6/15 = 2/5$ and $4/10 = 2/5$, show that the two given fractions represent the same rational number. Another way of showing them equal would be to equate each fraction to one whose denominator is the product of the given ones. That is

$$6/15 = 60/150 \text{ and } 4/10 = 60/150.$$

In the first case we multiplied the numerator and denominator by 10 and in the second case by 15. If we do this using letters it is easier to see what the result looks like in general. Let the fractions be a/b and c/d. Then

$$(a/b) = (ad/bd), \text{ and } (c/d) = (cb/db).$$

Now $bd = db$, by the commutative property and $cb = bc$. Thus the fractions (that is, the rational numbers which they represent) will be equal if $ad = bc$ and if $ad = bc$, the fractions will be equal.

Exercises C

1. Prove in two ways that each of the following pairs of fractions represent the same rational numbers: 6/21 and 10/35, 9/12 and 21/28.
2. In the second method above we tested the equality of the two fractions by making the denominators equal. Could we

have the numerators equal instead? If so, what would the conclusions have been?

3. Use the conclusion that $a/b = c/d$, if $ad = bc$ to make the following as true or false

   (a) $(2/3)=(20/30)$ (b) $(1/10)=(100/1000)$ (c) $(5/6)=(51/61)$
   (d) $(4/5)=(7/8)$ (e) $(17/51)=(3/9)$ (f) $(1/2)\times(3/4)=9/24$

4. Reduce the following to lowest terms:

   (a) $100/300$ (b) $50/250$ (c) $8/36$
   (d) $96/108$ (e) $121/143$ (f) $1924/2036$

5. Show that $(4/7)\times(7/4)=1$ and that $(9/17)\times(17/9)=1$.

6. Show that $(a/b)(b/a)=1$. The fraction b/a is called the reciprocal of a/b.

7. Write the reciprocals of the following numbers:

   (a) $2/3$ (b) $10/11$ (c) $29/3$ (d) $99/100$ (e) 10

8. Which of the following are correct and which are wrong? Give the reasons for your answers:

   $3/(2+6)=1/(2+2)$ $3/(2\times6)=1/(2\times2)$
   $3/(6+12)=1/(2+4)$ $3/(6\times12)=1/(2\times4)$

   Just showing that the numbers are equal or not equal is not enough. The proper or wrong use of the fundamental properties of rational numbers and natural numbers should be stated.

9. Is the following statement true for every rational number: Given a rational number, the reciprocal of its reciprocal is the given number.

5. Division by zero. So far we have specified that both the numbers appearing in a fraction must be natural numbers. Why did we fail to mention fractions like 3/0? We know that 3/2 was defined so that $(3/2) \times 2 = 3$. So 3/0 would have to be defined, if at all, so that $(3/0) \times 0 = 3$. This would seem peculiar since we know that any natural number multiplied by zero is zero. But we still might not be disturbed by this. Suppose we carry it a little further. Then $\left[(3/0) \times 0\right] \times 3 = 3 \times 3 = 9$. But $(3/0) \times (0 \times 3) = (3/0) \times 0 = 3$. Hence the assumption $(3/0) \times 0 = 3$ either leads to $9 = 3$ or that $\left[(3/0) \times 0\right] \times 3$ is not equal to $(3/0) \times (0 \times 3)$ which would deny the associative property. Our only choice is then to exclude zero denominators.

Exercises D

1. Should 0/3 be included among the rational numbers? Why? If it should be included, what number would it have to be equal to?
2. Use the argument in the paragraph above to show a contradiction we would reach if 4/0 were defined to be 4.
3. How many times is each of the following contained in 1?

   (a) 1/2 (b) 1/3 (c) 1/10

   (d) 1/100 (e) 1/1000 (f) 1/1000000
4. Does the question: "How many zeroes are contained in 1?", have any meaning? Why?

5. Could 0/0 be admitted to the family of rational numbers without running into trouble? Why?

6. Division of rational numbers. We have seen that it is easy to multiply two rational numbers. How can we divide them? Suppose we consider the quotient: $(3/5) \div (4/7)$. There are two ways to find this quotient. In the first place, we know that 2/3 is that number which when multiplied by 3 gives 2. Hence if we are to find the quotient $(3/5)/(4/7)$ we must search for a number which, when multiplied by 4/7 gives 3/5. In other words, we want to start with 4/7 and by multiplying by a properly chosen rational number, arrive at 3/5. If x stands for the number we are seeking, then

$$(4/7) \cdot x = 3/5$$

We can first multiply 4/7 by 7/4 to make the product 1, then by multiplying by 3/5 to obtain the product $1 \times 3/5 = 3/5$. Hence the number we are seeking is $(7/4) \times (3/5) = 21/20$.

We see $(3/5) \div (4/7) = 21/20$. To check, we find $(4/7)(21/20) = 84/140 = 3/5$. 21/20 is the number which multilied by 4/7 gives 3/5.

We can think of the quotient $(3/5) \div (4/7)$ as a quotient of two rational numbers, or as a single fraction with the rational number 3/5 as the numerator and the rational number 4/7 as the denominator, $\frac{(3/5)}{(4/7)}$

Then another way to get the same result is to notice that we can make the denominator of the given fraction 1 by multiplying numerator and denominator by the reciprocal of 4/7, that is, by 7/4. Then we have

$$(3/5)\Big/(4/7) = (3/5) \times (7/4)\Big/(4/7)(7/4) = (3/5) \times (7/4)\Big/1$$

$$= (3/5) \times (7/4) = 21/20.$$

How would you formulate this in words? This shows that we have in the rational numbers a system which has one advantage over the natural numbers. The natural numbers are not closed under division, that is, the quotient of two natural numbers is not always a natural number. But the rational numbers are closed under division except by zero, since the quotient of any two rational numbers is a rational number.

Exercises E

1. What is 3 divided by one-half? Find the result using division of fractions. Show how the same result could be obtained without dividing fractions.
2. Find the quotients:

   (a) $(3/2) \div (9/4)$ (b) $(9/4) \div (7/6)$ (c) $(3/2) \div (7/6)$

   (d) $(5/6) \div 2$ (e) $3 \div (3/4)$ (f) $(10/11) \div (2/5)$
3. Find the quotients:

   (a) $(3/2) \div \left[(9/4) \div (7/6)\right]$ (b) $\left[(3/2) \div (9/4)\right] \div (7/6)$
4. Is division of rational numbers associative?
5. Find the quotient:

$$(a/b) \div (c/d)$$

6. State the results in Exercise 5 in words.

7. Addition of rational numbers. We have seen how to multiply rational numbers. How do we add them? If the denominators are the same, it is easy. For instance

$$3/7 + 2/7 = 3 \times (1/7) + 2 \times (1/7) = (3 + 2) \times (1/7)$$
$$= 5 \times (1/7) = 5/7.$$

We just assume that the distributive property will hold and define addition accordingly. We could express this in terms of letters:

$$a/c + b/c) = (a+b)/c.$$

When the denominators are equal we add the numerators.

Suppose we have two rational numbers whose denominators are not equal. Then we can make the denominators equal by multiplying numerator and denominator by appropriate natural numbers and then add the numerators. Suppose we wish to add 2/7 and 3/5. Then we have

$$2/7 + 3/5 = 10/35 + 21/35 = (10 + 21)/35 = 31/35.$$

We chose the 35 as the denominator since it had to be a multiple of 7 and of 5 and the smallest such number is 35. Suppose on the other hand, we were to add 3/4 and 7/10. Here our denominator must be a multiple of 4 and also of 10. While 40 satisfies these conditions, 20 is a smaller number which does. Thus the numbers could be written

$$3/4 + 7/10 = 15/20 + 14/20 = 29/20.$$

You may prefer to write this in column form as

$$\begin{array}{rl} 3/4=15/20 & \text{or } 15(1/20) \\ +\ 7/10=\underline{14/20} & \text{or } +\underline{14(1/20)} \\ 29/20 & \phantom{\text{or }} 29(1/20)=29/20 \end{array}$$

Exercises F

1. Find the sums:

   (a) 7/8 + 3/8 (b) 3/5 + 6/5 (c) 7/8 + 3/16

   (d) 7/8 + 3/5 (e) 7/8 + 9/21 (f) 7/8 + 11/20

   (g) 11/12 + 3/24

2. Find the value of the following:

   (a) 1 ÷ (1/3 + 1/5) (b) (3+5) ÷ (1/3 + 1/5)

3. We defined addition so that the sum of two rational numbers having equal denominators was obtained by adding the numerators. Is the sum of two rational numbers having equal numerators, obtained by adding their denominators; that is, is 5/7 + 5/3 = 5/10? Give reasons.

4. Find the value of (8/13) - (2/7).

5. How would you subtract one rational number from another?

6. If possible using non-negative rationals, subtract from 7/8 the following:

   (a) 1/4 (b) 2 (c) 3/4 (d) 8/9 (e) 13/16

7. Find the value of $(7/4)\left[(6/7) + (9/8)\right]$

8. Find the values of:

   (a) 0/3 + 7/8 (b) 0/a + b/c

   (c) (0/3) x (7/8) (d) (0/k) x (a/b)

9. Find: $(a/b) \div (c/d)$

10. Why should finding the least common multiple of the denominators of two fractions be useful in finding the sum of two rational numbers?

11. Is every whole number a rational number? Why?

12. Two fractions $0/a$ and $0/b$ are equal when a and b are any natural numbers since $0 \times b = a \times 0$. Also $0/a$ is zero since $(0/a) \times a = 0$ and $0 \times a = 0$. Show that if the product of two rational numbers is zero, one or both must be zero. (Of course, you may assume that this property holds for the whole numbers.)

8. Summary of the properties of the non-negative rational numbers. It is probably worth while to list the properties which we have found so far. The rational numbers are represented by "ordered pairs" of numbers the first of which is a whole number and the second of which is a natural number. We call them "ordered pairs" since the order in which they are written is important; that is 3/4 is not the same number as 4/3. We use the solidus (the name for the slanting line) to separate them. But we could write 3,4 or $\frac{3}{4}$ or $3 \div 4$ just as well. We defined equality, sum and product and they have the following properties:

1. (closure) The product and sum of any two rational numbers are rational numbers.

2. (existence of identity number for addition and multiplication). The number 0 is a rational number and has the property that $0 + r = r$ for any rational number, r; zero is the identity number for addition. The number 1 is a rational number and has the property that $1 \times r = r$ for any rational number, r; one is the identity number for multiplication.

3. Addition and multiplication are associative.

4. Addition and multiplication are commutative.

5. The distributive property holds.

6. The quotient of any two rational numbers is a rational number if the divisor is not zero.

7. If the product of two rational numbers is zero, one or both must be zero.

8. Zero multiplied by any rational number is zero.

Exercise G

Which of these properties are also properties of the set of whole numbers?

9. Ordering of rational numbers. Let us first review a few facts about the whole numbers. We are familiar with the notation $7 - 5 = 2$. This is just another way of writing $7 = 5 + 2$. In words, $7 - 5$ is the number which, when added to 5 gives 7. Now $5 - 7$ is not a whole number since there is no whole number which we can add to 7 to get 5. Similarly,

18 - 10 is a whole number but 10 - 18 is not. In general, one natural number minus another natural number is a natural number only if the first is greater than the second. There is a notation for this: $7 > 5$ or $18 > 10$ means "7 is greater than 5" or "18 is greater than 10". We could also say "5 is smaller than 7" or written $5 < 7$; or "10 is smaller than 18" written $10 < 18$. This could be written in terms of letters as follows:

$b - a$ is a natural number if $b > a$, that is $a < b$.

These same symbols of inequality are useful in dealing with rational numbers. Suppose we wish to compare 1/3 and 2/7; which is greater? One way of doing this would be to find their decimal equivalents; this we shall do in the next section. The second way, which is probably simpler, is to replace the pair of fractions by a pair with the same denominator just as if we were going to add them. That is, $1/3 = 7/21$ and $2/7 = 6/21$. Since 7 is greater than 6, this shows that 1/3 is greater than 2/7. Another way to look at it is to see that $1/3 - 2/7 = 1/21$ which is the quotient of two natural numbers. In general, one rational number is said to be greater than a second rational number if the first minus the second is the quotient of two natural numbers; another way to say it would be: one rational number is greater than a second if one can add the quotient of two natural numbers to the second to get the first.

Notice that we have used "the quotient of two natural numbers". Why did we not just say "rational number"? One reason is that zero is a rational number and if their difference were zero, they would be equal. Also we shall later be considering negative rational numbers, and we wish to exclude them from our definition.

Exercises H

1. Associate 7/10, using the appropriate symbol $<, =, >$ with each of the following:

   9/10, 1/10, 1/2, 3/5, 3/4, 3/7, 0, 21/30, 7/5, 7/6, 7/8, 7/9, 7/11, 7/12

2. If two rational numbers have the same denominator, the larger rational number has the larger numerator. If two rational numbers have the same numerator show that the larger rational number has the smaller denominator.

3. Write the following rational numbers in increasing order:

   8/9, 18/19, 3/4, 5/6, 25/27

4. Write all the fractions between 0 and 1 whose denominators are 7 or less, in increasing order of size. There are a number of interesting properties of this set of numbers. Can you discover them?

5. If a, b, c, d are natural numbers show that $(a/b)>(c/d)$ if $ad > bc$. Show also that if $ad < bc$ then $(a/b)<(c/d)$. How could this be used to shorten the computation above?

6. If the diameter of a circle is 1, it is shown in geometry that the number of units in the circumference is a number designated by ○ and whose value to five decimal places is 3.14159. The rational number most often used as an approximation for this number is 22/7. Another approximation used by the Babylonians is 355/113. Which of these fractions is the greater and which is closer to π ?

7. If a and b are two integers, then just one of the following relationships holds: $a > b$, $a = b$, $a < b$. Show that the same statement may be made when a and b are rational numbers.

8. (Hard) Let r and s be two positive rational numbers with r ○ s. Show each* of the following for two pairs of values of r and s. For example, use $r = 1/3$ and $s = 2/5$.
    - a. $r < \left[(r + s)/2\right] < s$.
    - b. $1/s < \left[(1/r + 1/s)/2\right] < 1/r$.
    - c. $r < \dfrac{2}{1/r + 1/s} < s$.
    - d. if $r = a/b$ and $s = c/d$, then $r < (a+c)/(b+d) < s$.

9. What part or parts of exercise 8 show, if they are true in general, that between any two rational numbers there is another rational number?

* The notation in the parts of the exercise perhaps need further explanation. We write $2 < 4 < 7$ to mean "2 is less than 4 and 4 is less than 7" or, more briefly, "4 is between 2 and 7 and equal to neither". The corresponding meaning would be used for rational numbers.

10. (Very hard) Show that the inequalities of exercise 8 hold for all positive rational numbers r and s.

10. Decimal equivalents of rational numbers. We saw above that one way to compare the size of 1/3 and 2/7 was to compare their decimal equivalents. To limber up our pencils and our minds, let us start by finding a few decimal equivalents.

Exercises I

1. Find the decimal equivalents to ten places of each of the following: 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9, 1/10, 1/11. Draw a line with these numbers marked off on it.
2. Point out any patterns which you see in the decimal equivalents which you have just calculated. In particular was there any stage at which you could write down the answer without carrying the actual division farther? Which of the decimal equivalents were exact?

First of all let us look at these decimal expansions which are exact, that is, which end with a string of zeroes. We had 1/2 = .5, 1/4 = .25, 1/5 = .2, 1/8 = .125, 1/10 = .1. This kind of decimal is sometimes called a terminating decimal since it stops. Instead of using the decimal notation we could have used fractions. Then we would have 1/2 = 5/10, 1/4 = 25/100, 1/5 = 2/10, 1/8 = 125/1000, 1/10 = 1/10.

3. Express each of the following decimals as a quotient of two integers where the denominator is a power of ten, that is, one of 10; 100; 1000; 10000; etc: 15.78, 1.7893, .0012.

4. Do you believe that every terminating decimal can be expressed as the quotient of two integers in which the denominator is a power of ten? Why?

5. Express each of the following as a decimal: 156/1000, 57/10000, 789/100, 3589/10. Do you believe that any quotient of two integers in which the denominator is a power of ten, can be expressed as a terminating decimal? Why?

6. What connection is there between the answers for exercises 4 and 5 above? Collect these results into a single statement if you can.

7. The fraction 1/8 could be written as a terminating decimal, as we saw above, because it can be written as a fraction whose denominator is 1000, namely, 125/1000. 1/25 could be written as a terminating decimal because it is equal to 4/100. Is there any way that one can determine when a rational number has a terminating decimal without converting it to a fraction with a power of ten as denominator?

We saw from exercise 7 and other examples that if the fraction a/b is to have a terminating decimal it must be equal to a fraction c/d where d is a power of 10. Now if a/b is in lowest terms, it can be equal to c/d only if b divides d. In other words b must divide a power of 10. For example, 8 must divide 1000 so that 1/8 can be equal to 125/1000, 25 must divide 100 so that 1/25 is equal to 4/100.

8. If a rational number is a divisor of a power of 10 what can you say about its prime factors? (The teacher should recall to the student what is meant by "prime factors").
9. If a number has no prime factors but 2 and 5, must it be a divisor of a power of ten? Illustrate your conclusion with several examples.

So we can summarize what we have found so far by the following statement: If a rational number has a terminating decimal equivalent and if the fraction is in lowest terms, then the only prime factors of the denominator can be 2 and 5. Conversely, if the denominator of a fraction in lowest terms has no prime factors but 2 and 5, then its decimal equivalent terminates.

11. Repeating Decimals. In the first exercise of the preceding section we found that there were several fractions whose decimal equivalents did not terminate: 1/3, 1/6, 1/7, 1/9, 1/11. These do not terminate since the denominators have factors different from 2 and 5. Next we look into these in more detail.

One way to write a decimal equivalent for 1/3 would be .3333$\underline{3}$... where the line under the 3 and the three dots afterward indicate that no matter how far out one carries the division there will be just a series (or string) of threes. Similarly 1/11 could be written .0909$\underline{09}$... where it is the pair of digits 09 which repeat as far as the division is carried out. Also 4/3 can be written 1.33$\underline{3}$... . The fraction 1/7 has a repeating portion of six digits: .$\underline{142857}$... . Such decimals as these are called *repeating decimals* (sometimes, periodic decimals). That is, a decimal is called a repeating decimal when from a certain point on some sequence of digits repeats and continues to repeat no matter how far the division is carried out. Notice that 1.33$\underline{3}$... is a repeating decimal even though the initial digit is not 3. Similarly, 14.235$\underline{235}$... is a repeating decimal.

These decimals which we have found for 1/3, 1/7, etc., do not give the exact value for the fraction no matter where one cuts them off but the farther one goes, the closer is the decimal in value to the number. For instance 1/3 - .3 = 1/3 - 3/10 = 1/30, 1/3 - .33 = 1/3 - 33/100 = 1/300, 1/3 - .333 = 1/3 - 333/1000 = 1/3000 and so on. The results of Exercise 2 below show similar results for the expansion of 1/7. For instance, .142857 is equal to 142,857/1,000,000 but this is not equal to 1/7 since 7 times 142,857 is 999,999 which is just short of 1,000,000. However 1/7 - .142857 = 1/7000,000 which is a very small number.

Exercises J

1. Sometimes one writes 1/3 = .33 1/3. What does the second 1/3 stand for? Is it the same as the first 1/3? Is the following true: 1/3 = .333 1/3? If so, what does the second 1/3 stand for here?
2. What would one have to add to .142 to make it exactly equal to 1/7? What would one have to add to .1428 to make it exactly equal to 1/7?
3. Multiply each of the following by 3: .3, .33, .333, .3333. By how much does each of your results differ from 1? What connection is there between your answers here and exercise 10 above?
4. Find the decimal equivalents for each of the following fractions. (Do not be discouraged at the size of the denominator in some cases. The process is shorter sometimes than for smaller denominators.) Carry out the division to the point where the decimal terminates or begins to repeat.

   (a) 3/8 (c) 15/37 (e) 41/333 (g) 1/13
   (b) 5/44 (d) 7/125 (f) 4115/33,333 (h) 1/17 .
5. In the decimal equivalents for 1/3, 1/7, 1/13, 1/17 do you see any connection between the denominator and the number of digits in the repeating part of the decimal?

6. Do not carry out the division for 5/413 but guess whether or not the decimal will terminate or repeat. Give reasons for your guess. How far might you have to carry out the division to show your guess to be correct or false?
7. How many different remainders would it be possible to have in dividing a number by 727? What would a remainder have to be if the decimal terminates?

12. Rational Numbers Equivalent to Repeating Decimals.

It is a remarkable fact that the decimal equivalent of every rational number either terminates or is a repeating decimal. You may have guessed this already. To see why it is so, consinder first a few divisions. First take 4/15 (this division is to be written out). Here the remainder after two divisions is the same as after three and the process just repeats itself. Consider the decimal for 2/7 (this division is to be written out). Here the first remainder is 4 and the remainder after six divisions is 4, which means that the series repeats. Consider 575/17 (this division to be written out). The first remainder is 6, the second is 14, the third is 4 and the fourth 6. It does not at this point begin to repeat since the first 6 occurred before zeroes were adjoined. But as soon as a third 6 occurs as a remainder the decimal will begin to repeat.

Thus in finding the decimal equivalent of any rational number either the decimal will terminate or, from a certain point on one must continue to adjoin zeroes to the dividend. If, after zeroes are adjoined, two remainders are the same the decimal begins to repeat and continues to do so. Why must two of these remainders be equal? The first exercise should be a help in reaching the answer to this question.

In division by 17, the only possible remainders would be 0, 1, 2, 3, ... , 15, 16. If a remainder is zero, the decimal will terminate. From what we have shown above, this would not happen if the fraction were in lowest terms since the denominator has factors other than 2 and 5. If the decimal does not terminate, there would be only 16 possible remainders. Suppose in finding the decimal equivalent of 1/17 the first sixteen remainders were all different (we saw above that this was indeed the case). Then the next one would have to be a remainder that had occurred before. Similarly, in dividing by 37 there would be 36 possible remainders not more and if the first 36 were all different, the next one would have to be one which already occurred. Actually the first three remainders in computing 42/37 are 5, 13, 19 and the fourth remainder is 5 again. Since the first remainder occurred after a zero was adjoined to the 42, the decimal repeats from the fourth remainder on. It is $1.135\underline{135}$ ... . Hence it is not necessary that all the remainders occur before repetition.

But we can be sure in every case that the largest possible number of digits in the repeating part of any repeating decimal is one less than the divisor.

This discussion shows that every rational number has a decimal equivalent which either terminates or is a repeating decimal.

So far we have considered converting rational numbers into their decimal equivalents. Suppose we have a repeating decimal: .12$\underline{12}$ ... . Can we find a rational number which it represents? Before trying this let us go back to one which we already know and develop a method for dealing with it so that we may apply it to the case at hand.

Consider the decimal: .33$\underline{3}$ ... . Let the letter n stand for this number. Then ten times this number, that is 10n will be 3.33$\underline{3}$ ... . That is, we have

$$10n = 3.33\underline{3}\ \ldots$$
$$n = .33\underline{3}\ \ldots$$

If we subtract n things from 10n things we have 9n things. (This can also be seen from the distributive property: $10n - n = (10 - 1)\,n = 9n$). And .33$\underline{3}$ ... subtracted from 3.33$\underline{3}$ ... is 3.000. Hence we have $9n = 3$. But, using our notation for a rational number, we see that this means $n = 3/9$ which is equal to 1/3. This is a complex way of showing that 1/3 has the decimal equivalent given above but it is useful to look at this process since it will apply for more difficult decimals.

Now let us return to $.1212\underline{12}$ ... - $.1212\underline{12}$ ... . Try instead $100n = 12.12\underline{12}$ ... . Then $99n = 12.000$ and $n = 12/99$ which reduces to 4/33.

We do not attempt to give a formal proof that every repeating decimal represents the quotient of two integers, that is, a rational number, but working the exercises which follow should be evidence in that direction.

Exercises K

1. Find the rational number whose decimal equivalent is $.1212\underline{12}$············. Find the rational number whose decimal equivalent is $.121121\underline{121}$·······.
2. Express each of the following in the form a/b where a and b are integers: $.3434\underline{34}$... , $1.3434\underline{34}$... , $13.4343\underline{43}$... , $.567\underline{567}$... , $1.23412341\underline{234}$... , $5761.23123\underline{123}$... .
3. Can you formulate any rule for determining what n is to be multiplied by in dealing with such repeating decimals?
4. Look again at the number of digits in the repeating parts of the decimal equivalents for 1/3, 1/7, 1/11, 1/13, 1/17. We have seen above that the number of digits in the repeating part cannot be as great as the denominator. Can you discover any sharper relationship?

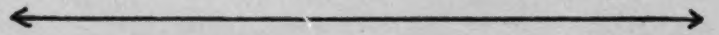
UNIT VI

NON-METRIC GEOMETRY

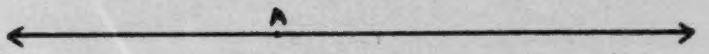
Sets and the Intersection of Sets

In your study of arithmetic you have learned about properties of number and properties of the operations (addition, subtraction, multiplication, division) on numbers. We are now going to consider the ideas of point, line, and space. A study of these ideas is called geometry.

We shall think of a line as being a representation of certain things, such as the edge of the desk or the edge of a ruler. We use the term line to mean straight line. A line will be represented in a sketch in this way:



The arrows are used here to show that the drawing represents a line extending in both directions without end. We use notation in the next sketch to label a particular point, A, on the line.



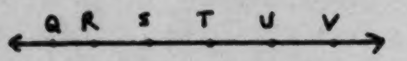
In studying lines we use the ideas of point and set. A set consists of all the things which have some characteristic, or property, in common. For example, the pupils in this class are a set. Each pupil is a member, or element of the set. The even numbers form a set. Three elements of the set of even numbers are 2, 16, and 38. The set of even numbers smaller than 10 are 2, 4, 6, and 8. This set has just four elements. The persons who have been president of the United States are a set. Thomas Jefferson is one element of this set, and President Eisenhower is another element. A set may have no elements, and in that case we call it the "empty set," or the "null set." Probably the set of pupils in your class who are less than eight years old is the empty set.

A line may be thought of as a set of points. Any point on the line is an element of this set.

Two (or more) sets which have common elements are said to have an *intersection*. The intersection is also a set. It is the set of all elements in *both* sets. The intersection set of the set of even numbers and the set of whole numbers from 1 to 9 contains the numbers 2, 4, 6, and 8 since these numbers are in *both* sets. In this room we have a *set of boys* and a *set of people who have a birthday in May*. The boys who have a birthday in May are in the intersection set. If no boy in the class has a birthday in May the intersection of the two sets is the empty set.

EXERCISES

1. Write the elements of the set whose members are
   a. The whole numbers greater than 17 and less than 23.
   b. The first names of members of your family.
   c. The cities over 100,000 in population in your state.
   d. The members of the class less than 4 years old.
   e. The names of special kinds of quadrilaterals.
2. Write 3 elements of each of the following sets:
   a. The odd whole numbers.
   b. The whole numbers divisible by 5.
   c. The set of points on the line below, some of which are labeled in the figure:

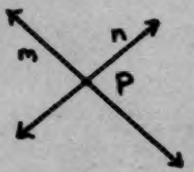


3. Give the elements of the intersections of the following pairs of sets:
   a. The whole numbers 2 through 12 and the whole numbers 9 through 20.

b. The digits in your telephone number and the odd numbers.

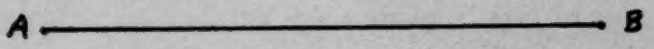
c. The members of the class and the girls with blond hair.

d. Set of points on line m and set of points on line n .



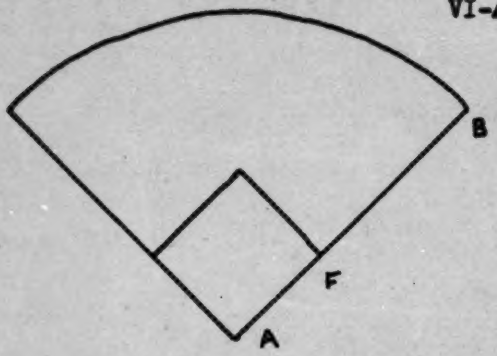
Line Segments and Half-Lines

When we represent a line by a sketch on the blackboard or a piece of paper our sketch shows only a part of the line, or a line segment.



A line segment is determined by 2 points on the line, in this sketch A and B, which are called the endpoints of the segment. The segment, AB, is a set of points. If the edge of a book is thought of as a line segment, the endpoints are the two corners of the book on that edge. An edge of a foot ruler represents a line segment and the endpoints are usually marked 0 and 12.

On a baseball diamond, we may think of home plate as a point, and first base as a point. Then the part of a line from home plate to first base is a line segment of which the endpoints are homeplate and first base. When this line segment is extended from first base into the outfield, it is called the right-field foul line. A ball which is hit left of it is "fair" and a ball which is hit right of it is "foul." In baseball, the foul-line is a line segment extending from home plate, as one endpoint, through first base into the outfield to the ballpark fence. The point of intersection of the foul-line with the ballpark fence is the other endpoint of this line segment (foul-line). First base, thought of as a point, is a member of the set of points on the line segment (foul-line).



A represents home plate.

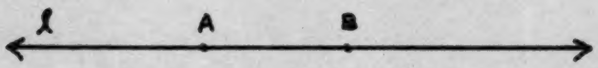
F represents first base.

B represents the intersection of the foul-line and the ballpark fence.

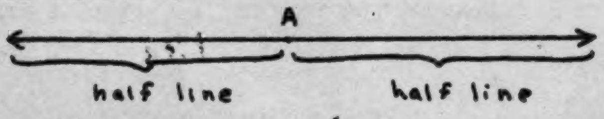
In the figure, AF and AB represent the line segments described above.

In our idea of a line we agree that there is always a third point between any 2 points of the line. This third point is said to be between the 2 points. In the sketch above, F is said to be between A and B. On every line segment there is always a point of the segment between the endpoints. Recall that with numbers there is always a third rational number between any two given rational numbers. For example, between $\frac{1}{4}$ and $\frac{1}{2}$ is $\frac{3}{8}$, and also $\frac{7}{16}$.

A line segment can always be extended in both directions without limit and this extension of the line segment is called a line, or straight line, as we have seen. The figure shows the line segment AB extended to form a line. A and B and all other points of set AB are elements of the set of points on the line ℓ.



In the figure below, consider the set of points on the line ℓ, and one element of the set, A. A divides the line or set, ℓ, into two half-lines -- the points on the line which are left of A in the figure, and the points on the line which are right of A. We say that A determines 3 sets on the line: the 2 half-lines and the set which contains only the element A. If we join A to one of the half-lines, we call the half-line with A adjoined, a ray.



The intersection of 2 rays on a line determined by a point on the line is the point itself, because the point is on both rays. The intersection set of two half-lines on a line is the empty set. In the example above about the baseball field, the segment from home plate to first base might be thought of as extending without limit in the direction of the outfield. Then the set of points from homeplate through the extended part of the line would be the elements of a half-line. If we join the point represented by home plate to this set, we have a ray.

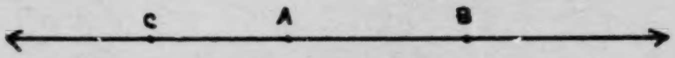
You see that "line" as we use it is merely the way in which we think of many situations in the world about us. The finest machine can never make a line in the mathematical sense, nor can we put the most fine needle point on a line without its touching many, many points of the line. We say that the sets of points on a line and the line itself are "ideal" ways in which we can describe all kinds of situations in our space.

We shall refer to

(a) the sets of points on line ℓ , or the set ℓ .

(b) the set of points on half-lines or the set of points on rays determined on ℓ by A. To distinguish the two half-lines, or the two rays, we may refer to the half-line from A on B, or the half-line from A on C.

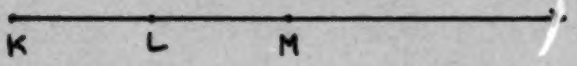
(c) the set of points on the segment AB, or the set AB.



EXERCISES

1. If you think of Main Street in your town as a line segment, what are the endpoints of the line segment? Assume Main Street is straight.
2. Use a ruler to draw a line segment on a piece of paper. Label the endpoints of the segment A and B.
   a. Label as D a point of the set AB.
   b. Label as E a point of the set DB.
   c. Is E a point of the set AB?
   d. Is E a point of the set AD?
   e. Label F of the set AD.
   f. List the segments in your sketch and list also each of the labeled points which are on any segment.
3. a. Extend a segment AB through B so that A is the endpoint of a ray.
   b. Label point C which is a point of the ray in a.
   c. In your sketch is C a point of a ray with endpoint B?
4. On a line label two points A and B.
   a. Name the four half-lines you have sketched
   b. List the different sets of points you have illustrated in the sketch.
   c. Label a point C between A and B.
   d. On what half-lines is C?
5. How could you associate the term "ray of light" with the way we are using the term ray?
6. The boundary line between the states of Illinois and Wisconsin may be thought of as a line segment. Think of the segment as extended in an eastward direction. Describe a ray and a half-line associated with this extended segment. (There are two answers.)

7. The figure represents rays with endpoints K, L, and M.



a. List 3 points of the ray with endpoint K.

b. List points on the half-line determined by K.

c. Describe two sets of points in the figure of which M is the intersection set.

d. Describe two sets of points in the figure of which the segment LM is the intersection set.

8. Draw a line, m, and on it mark a point R.

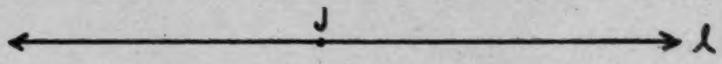
a. What is the intersection set of the two rays determined by R on m?

b. What is the intersection set of the two half-lines determined by R on m?

c. What is the intersection set of one of the half-lines in your sketch with each of the two rays in your sketch?

Lines on a Point

Just as we say "a point lies on a line" we may say "a line lies on a point."



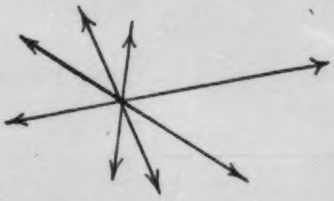
The figure is an illustration of the two sentences:

Point J lies on line ℓ

Line ℓ lies on point J.

If we think of the line determined by home plate and first base in a ballpark (home plate and first base are thought of as points) we may say that the line lies on home plate or that home plate lies on the line. Also, the line lies on first base, and first base lies on the line.

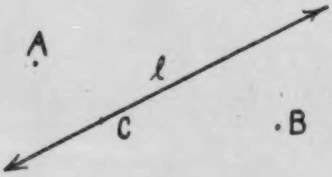
When we considered the set of points on a line, ray, half-line, or segment we thought only in terms of what might be called one dimension in geometry. All of our discussion in this chapter has been about points on a line (in one dimension) except for the ballpark illustration. Now we shall consider the set of lines on a point. You see that there are many lines on a point just as there are many points on a line. Let us first think of the set of lines on a point which are all in the same plane. The figure shows what we mean.

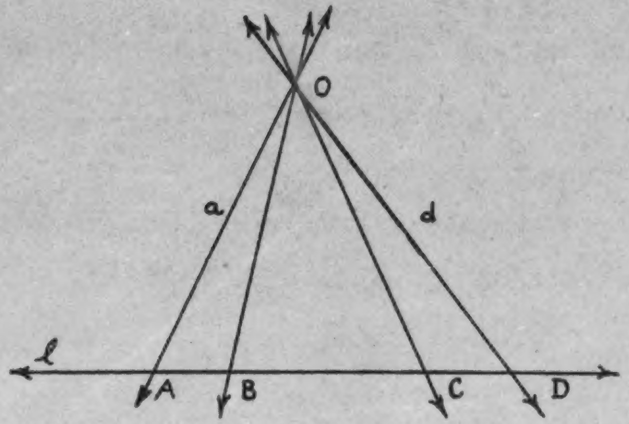


Perhaps this figure makes you think of the spokes of a wheel, a design in a church window, or an octopus with its "arms" all straight.

Like point and line, a plane is a mathematician's way of thinking about the "ideal" of a situation which we find all about us. We may think of a table top, the blackboard, or a soap film on a fine wire frame, as parts of planes. The geometry of the plane is geometry of two dimensions. The points on a plane are a set of points. The lines on a point are a set of lines.

A line divides a plane into two half-planes. In the figure below, point A is a point in one half-plane, and point B is a point in the other half-plane. Point C is on line ℓ, so it is not in either of the half-planes.





In the figure above we have represented a set of lines on a point O, and a set of points on line ℓ. We can see that the intersection of ℓ and a line on O is a point. For example, line a intersects ℓ on A. We can also see that any point on ℓ and point O determine a line on O. For example, D and O determine line d. To a line on O there corresponds a point on ℓ. To a point on ℓ there corresponds a line on O. A relationship like this is called a one-to-one correspondence. Do you see why this is a good term? To a point there corresponds a particular line, and to a line there corresponds a particular point.

EXERCISES

1. Give other examples of a set of lines on a point.
2. Give other examples of a plane (or parts of a plane).
3. Locate point A on a piece of paper. Sketch a set of lines on A.
4. Locate points A and B on your paper.
   a. How many lines may be drawn on point A?
   b. How many lines may be drawn on point B?
   c. Draw four lines on A, and four lines on B.
   d. The lines on A are a set of lines, and you have drawn four

elements of this set. Also, the lines on B are a ___?___ of lines, and you have drawn ___?___ ___?___ of this set.

e. Is any line of the set of lines on A also an element of the set of lines on B? If so, have you drawn it?

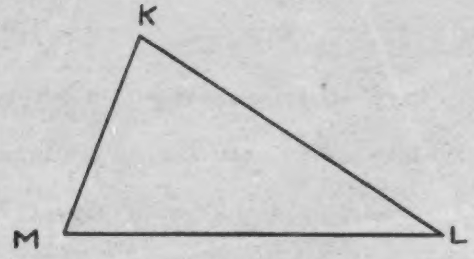
f. A line on A which is also a line on B is the ___?___ ___?___ of the set of lines on A and the set of lines on B.

5. Locate points R, S, and T on a piece of paper.

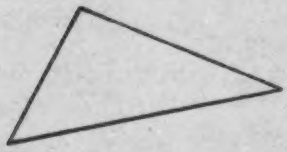
a. Sketch 4 of the elements of the set of lines on each point.

b. Include in your sketch the intersection set of each pair of sets in a.

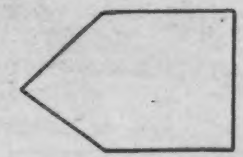
6. A _triangle_ is a figure composed of three points, not all on a single straight line, and the three line segments joining them in pairs. If the points are labeled K, L, and M, the three line segments are labeled KL, LM, and MK. K, L, and M are called the _vertices_ of the triangle. Each is a vertex. Since the points determine the triangle, we can call the triangle, triangle MKL, or more briefly, $\triangle$MKL. Which of the following are triangles?



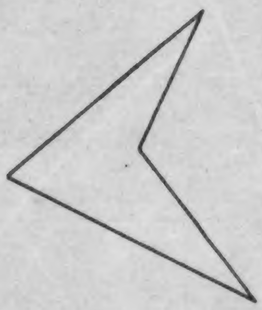
(a)



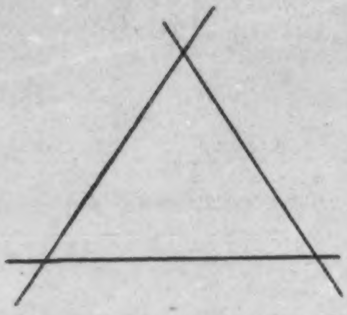
(b)



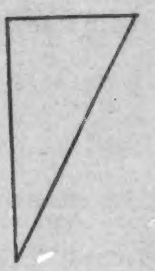
(c)



(d)



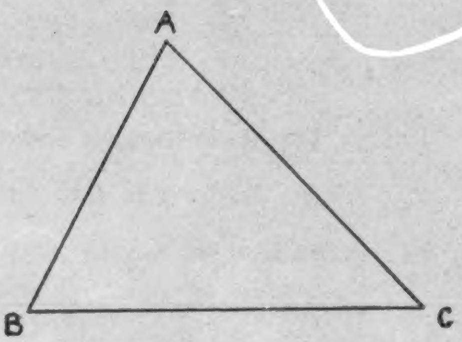
(e)



7. a. In the triangle what is the intersection set of AB and AC?

   b. Does a triangle contain any rays or half-lines?

   c. Draw triangle ABC and extend AB in both directions to obtain a line m. What is the intersection set of AB and m? What is the intersection set of m and the triangle?

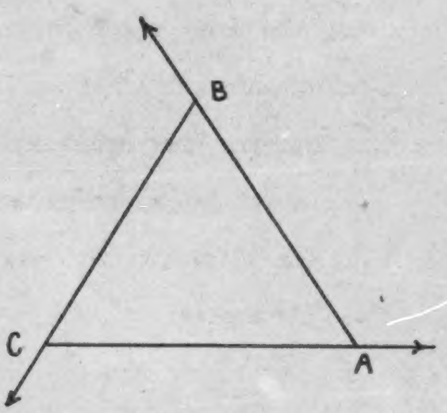


8. a. How many half-lines are shown in the figure?

   b. How many rays?

   c. On the ray through B of which A is the endpoint there are represented several sets of points as we have discussed them. Name as many as you can.

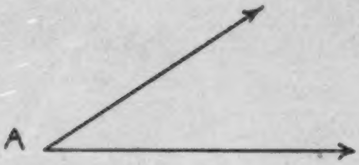
   d. Locate point D, different from A and B, so that it is in as many of the sets of c. as possible.



9. Give a definition of a quadrilateral which is similar to the definition of a triangle in Exercise 6.

10. We can think of a triangle as lying on a plane, or determining a plane. Can we conclude that a plane is determined by 3 points?

11. Describe this figure. The set of points on two rays with a common endpoint is called an angle. The figure represents an angle. The endpoint of the rays, A, is the vertex of the angle, and the rays are the sides of the angle.



12. In the figure for Exercise 8, find three angles. Remember the sides must be rays.

13. The opening between the rays which are sides of angle is the inside, or interior of the angle. Draw an angle and call its vertex A. Mark a point in the interior of the angle B. Mark a point which is not in the interior of the angle C.

14. Draw a set of lines on a point and then draw another line not on the point which intersects all of the lines of the set. Label points and lines in your sketch so that you can describe the one-to-one correspondence which has been established. (For example, point A might be made to correspond with line a.)

15. Is there a one-to-one correspondence between the pupils and the desks in your room? Why?

16. Describe a one-to-one correspondence between the natural numbers and the even natural numbers. (For example, 1 ↔ 2, 2 ↔ 4, 3 ↔ 6, etc.)

17. Describe a one-to-one correspondence between the points A, B, and C which determine a triangle and the line segments of the triangle. Can you do this in more than one way?

Closed Curves

In newspapers and magazines you often see graphs like these. These graphs are called curves and we shall consider the set of points

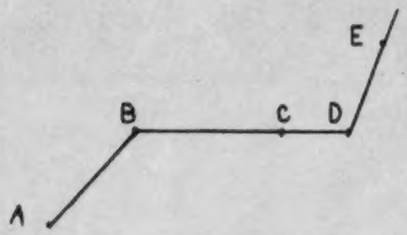


Figure I

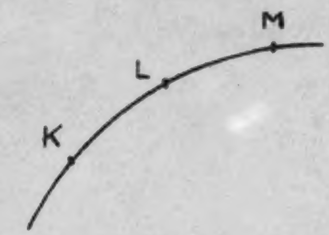


Figure II

on a curve. In Figure I, points A, B, C, D, and E are points on the curve, or the curve is on these points. Of course, between any two of these points are many other points. A, B, C, D, and E are elements of the set of points represented by the curve. K, L, and M are points on the curve shown in Figure II, and the curve in II is on the points K, L, and M. A line (or straight line) is a special kind of curve. Since we will call a straight line a special kind of curve, you see the term "curve" is used in mathematics differently from the way it is used in everyday language.

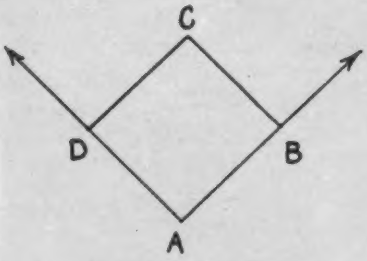


Figure III

Figure III represents the base lines on a ball field. All the points inside the curve made up of the segments AB, BC, CD, and DA are in the infield. If a ball is hit into the infield it is called a "fair" ball. If a ball is hit and then continues it could still be called a fair ball so long as it does not get outside of either of the two rays of which A is the intersection. (What do baseball rules say about this situation?) Since this curve clearly indicates when a ball is fair and when it is foul, it divides the ball field (and this extended as far as your imagination chooses) into an inside (fair territory) and an outside (foul territory). This idea of a curve which divides a plane into parts is a very useful one in geometry and some of its applications.

The curves in Figure IV, Figure V, and Figure VI are called simple closed curves.

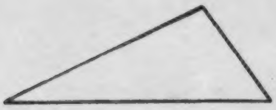


Figure IV

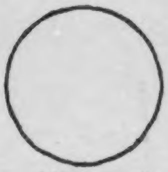


Figure V

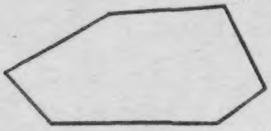


Figure VI

The curves in I and II are not closed curves. Think of all of these curves as plane curves or curves which lie on a plane. A simple closed curve has an outside and an inside and the curve is said to be the boundary of the outside and inside. If point A is inside a simple closed curve and point B is outside, it is not possible to pass in the plane of the curve from A to B without crossing the boundary. A simple closed curve divides a plane into 3 sets, the inside, the outside, and the boundary.

In one dimension a point divides a line into 3 parts or sets. In two dimensions a simple closed curve divides a plane into 3 parts or sets.

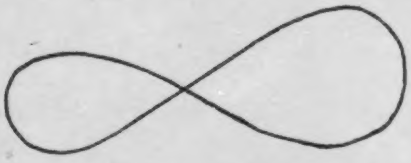


Figure VII

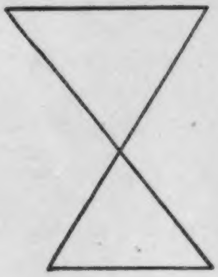


Figure VIII

Figure VII and Figure VIII are closed curves but we do not call them simple closed curves. Each of these curves appears to cross over itself.

In Figure IX, R and S are points in the intersection set of the two curves. R and S are points which lie on both curves. Both curves lie on R and S. T is a point inside the simple closed curve and U is a point outside the simple closed curve. R and S are points on the boundary of the simple closed curve.

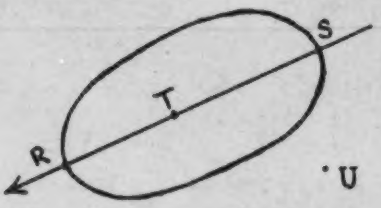
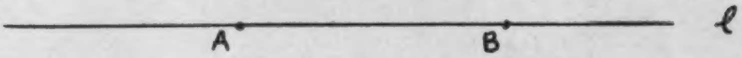


Figure IX

EXERCISES

1.

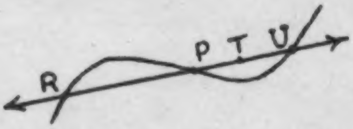


A and B of the set ℓ in the figure, may be thought of as bugs which can crawl only on line ℓ. Describe the simplest set on ℓ which will provide a boundary between A and B.

2. Think of A and B as bugs which can crawl anywhere in a plane. Describe the simplest set of points in the plane which will provide a boundary between A and B.

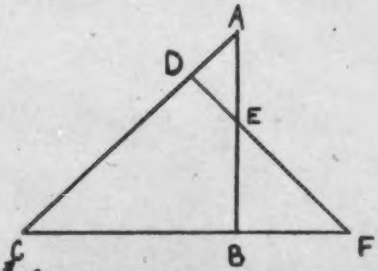
3. Sketch three different simple closed curves which are neither triangles nor circles.

4.

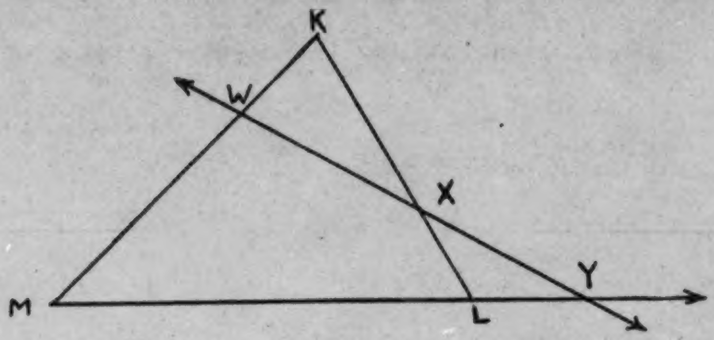


What is the intersection set of the S-curve and the line in the sketch?

5. What is the intersection set of the triangle ABC and the line segment DF?



Exercises 6 through 11 refer to this figure.



6. Describe the positions of W, X, and Y in the figure.

7. Name the triangles in the figure.

8. If possible, locate point I which is not inside any of the triangles in the figure.

9. If possible, locate point H inside any two of the triangles in the figure.

10. If possible, locate point G inside triangle MKL but not in any of the other triangles.

11. If possible, locate point J inside all of the triangles in the figure.

12. If possible, make sketches in which the intersection set of a line and a triangle is

    (a) the empty set　　(b) a set of one element

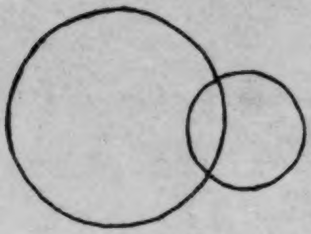
    (c) a set of two elements　　(d) a set of three elements

13. If possible, make a sketch in which the intersection set of a line and a circle is

    (a) the empty set　　(b) a set of one element

    (c) a set of 2 elements　　(d) a set of 3 elements

14. Consider the intersection, in all cases, of a line segment AB and a circle. How many situations must you consider? If A and B are both outside the circle, can the circle and the segment AB have just one point in common?

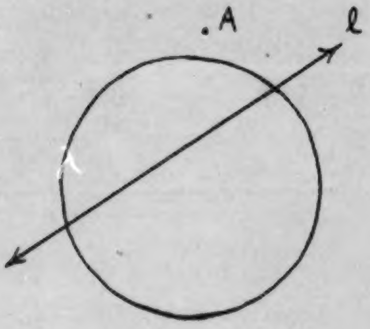
15. 

Sketch two simple closed curves like these for each part of this exercise.

a. Shade the intersection of the sets inside both curves.

b. Shade the intersection of the sets outside both curves.

c. Shade the intersection of the sets outside one curve and inside the other.

16. 

Draw a figure like the one shown.

a. Shade the intersection set of the set of points inside the circle and the set of points on the half-plane which contains A.

b. Shade the intersection set of the set of points outside the circle and the set of points in the half-plane which does not contain A.

c. Describe in your own words the set of points which are not shaded in either a. or b.

17. Sketch a circle and a quadrilateral for which the intersection set consists of

(a) 4 points (b) 6 points

(c) the largest number of points possible.

18. A line divides a plane into 3 parts or sets. What names would you suggest for these parts?

19. How would you define the inside and the outside of an angle? See the definition of angle, Exercise 11, page 12.

Planes on a Line

If you draw a line on the blackboard you have a representation of the mathematician's idea of a line lying on a plane. All points on the line lie on the plane. A line divides the set of points in a plane into 3 sets: the points on two half-planes and the points on the line. If A is in one half-plane and B is in the other, as in Figure 1, the intersection set of AB and the line is a point. What can you say if A and B are in the same half-plane? See Figure 2.

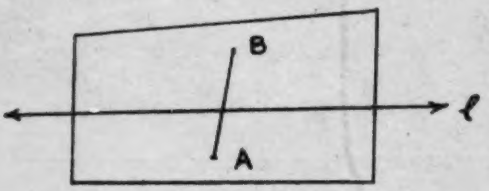


Figure 1

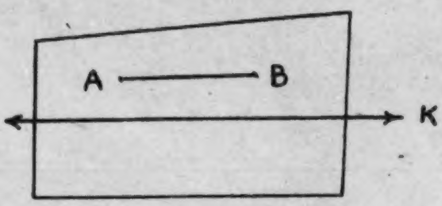


Figure 2

A simple closed curve also divides the plane on which it lies into 3 sets, as we have seen.

If a line lies on a plane, we can also say that the plane lies on the line. If A and B are points on a plane, then the plane lies on A and B.

There is a set of planes on a line. This idea is represented in Figure 3. If you fold a piece of paper and think of the crease as a line, you have a model of two half-planes on a line. Could you make a model of two planes on a line? Do you notice that in dealing with two planes our attention has been called to geometry in three dimensions?

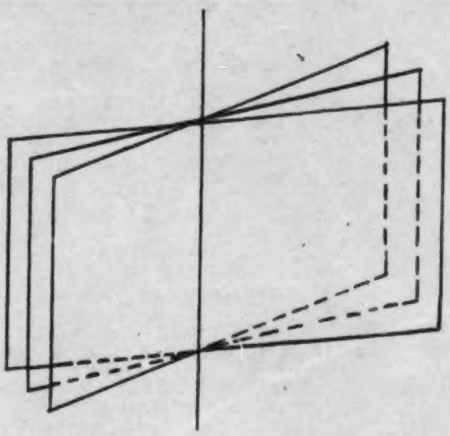


Figure 3

Figure 4 shows two planes on a line k and a second line m which intersects each of the planes in a point. The intersection set of two planes is a line. The intersection set of a plane and a line, which does not lie on the plane, is a point.

Lines like k and m in Figure 4, which have different directions and of which the intersection set is a null set, are called skew lines. Skew lines do not lie on the same plane. Do you see in the room a pair of line segments, which if extended would be skew lines?

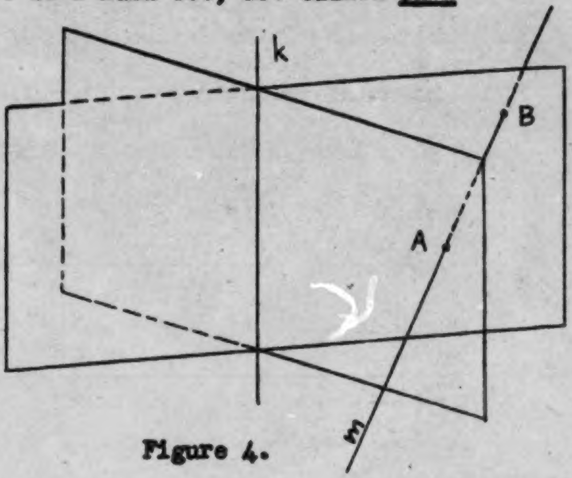
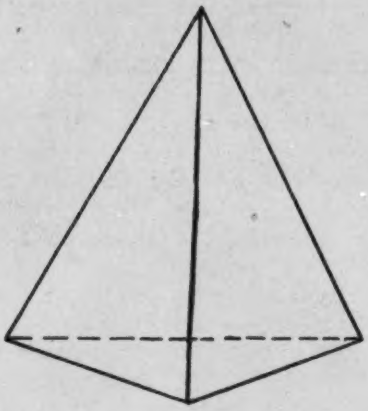


Figure 4.

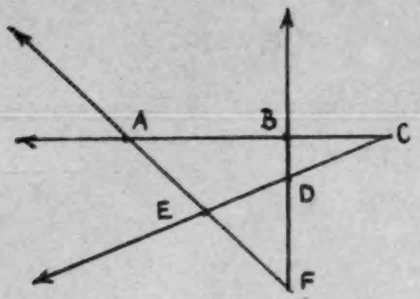
EXERCISES

1. Think of the front wall of the classroom, the ceiling, and the side wall as representing 3 planes.
    a. Describe the intersection set of the planes represented by the ceiling and the front wall.
    b. Describe the intersection set of the planes represented by the front wall and the side wall.
    c. Are there points common to all 3 planes?
2. Think of a case where points A and B are outside a plane and on opposite sides of a plane.
    a. Describe the intersection set of line segment AB and the plane.
    b. If A and B are outside a plane and on the same side of a plane, what is the intersection set of line segment AB and the plane?

3. Which pairs of lines determined by the intersection sets of the ceiling, the floor, and 4 side walls appear to be skew lines? (Skew lines do not have the same direction.)
4. Describe several examples of a set of planes on a line.
5. Describe several examples of two skew lines.
6. What is the intersection set of two skew lines?
7. A plane divides the set of points in space, into 3 parts. What are they? Suggest names for the parts.
8. Make a model of 3 planes which lie on the same line.
9. Describe the faces (sides, top, and bottom) of a chalk box and the edges in terms of sets and intersection sets. How many of the intersection sets are lines? How many of the intersection sets are points?
10. A chalk box is a solid with 6 faces. The figure shows a tetrahedron, which is a solid of 4 faces. Describe the faces and edges of a tetrahedron in terms of sets and intersection sets. How many of the intersection sets are lines? How many are points?



Special Figures



In the figure we have 4 lines and 6 marked points. On each line there are 3 of the points and on each point there are 2 of the lines. In projective geometry this figure is called a complete quadrilateral. Let us see how this figure can be thought of as a diagram for placing 6 people (the points) on 4 committees (the lines) so that all persons are treated exactly alike. Each person is on the same number of committees and no person is on the same two committees as any other person. Let us consider A and C for example.

A and C are both on 2 committees (and only 2).

A is on one and only one committee with B, C, E, and F.

C is on one and only one committee with B, A, E, and D.

On each of A's committees there are 2 other members shown.

On each of C's committees there are 2 other members shown.

An even more "democratic" committee diagram can be shown for 10 people and 10 committees. With this diagram we can place each person on 3 and only 3 committees and on each committee there will be 3 and only 3 of the 10 people. We might say that this represents real democracy in committee structure. The diagram of 10 points (people) and 10 lines (committees) is called the Desargues configuration. This is a very remarkable figure and is important in the study of projective geometry. Projective geometry arose from the interest of artists in perspective drawing. One of these artists was Leonardo da Vinci.

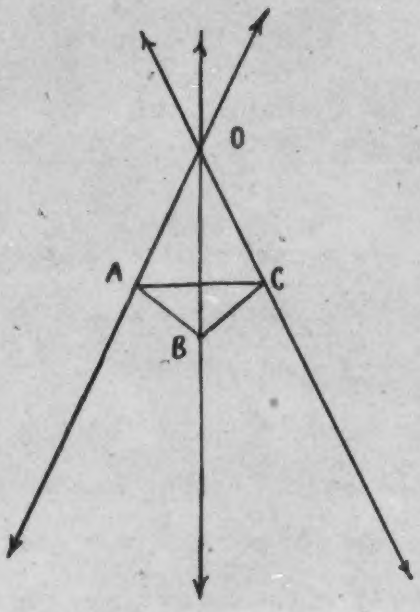
The exercises which follow are concerned with the Desargues configuration.

EXERCISES

1. In the figure there are three lines on a point O and on each of these lines is one of the vertices of the triangle ABC.

   a. Copy this figure on a separate piece of paper.

   b. Draw a second triangle A'B'C' so that A' is on OA extended, B' is on OB extended, and C' is on OC extended. (If possible choose the positions of A'B'C' so that AB extended intersects A'B' extended on your paper.)



   c. Label the points of intersection of the lines as follows:

      line on AB meets line on A'B' in D

      line on AC meets line on A'C' in E

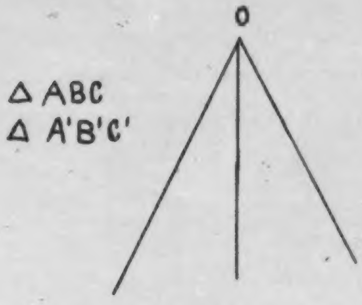
      line on BC meets line on B'C' in F.

   d. Do the points D, E, and F appear to lie on the same straight line? If not, try your drawing again. Draw the line on D, E, and F.

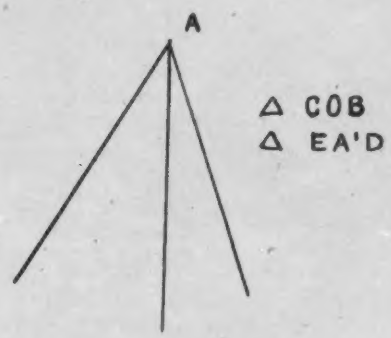
2. Make several drawings like that in Exercise 1.

3. In the completed figure in Exercise 1, you will notice that you have drawn 10 lines (or parts of them) and you have labeled 10 points. Examine your sketch to see if on each of the 10 lines lie 3 of the labeled points and on each of the 10 points lie 3 of the 10 lines you have drawn.

4. One of the remarkable aspects of the figure is that each point and each line play exactly the same role. For example, we may think of point A as the "beginning" point on which 3 lines lie (we used O for this point in Exercise 1). Then we can find on the 3 lines through A the vertices of the triangles COB and EA'D. A listing may help you see this.



lines on segments of
△s meet in D, E, F



lines on segments of
△s meet in C', F, B'

Shade the △s COB and EA'B in your Desargues figure. It is not necessary to mark any new point nor to extend any line to observe this.

5. Follow through the steps in Exercise 4, starting with point D. What are the vertices of the 2 triangles? In what points do the lines on the segments of the triangles intersect? Do these points lie on a line in your figure?

UNIT VIII

Informal Geometry I

Look around your classroom. The four walls, floor, and ceiling are parts of different planes. The planes intersect in lines, and some of the lines intersect in points. Perhaps your classroom has more than four walls, and perhaps the ceiling is made up of parts of several planes. The blackboard is probably in a different plane from the plane of the wall.

1. Find surfaces in the room and its furniture which are parts of different planes.
2. Find lines which are intersections of planes, and points which are intersections of lines.
3. Can you find three planes which intersect in the same line? Could you, if your room had a revolving door?
4. What planes can you find which do not intersect? What lines can you find that do not intersect?
5. Can you find some surfaces which are curved, not plane?
6. Look out of the window and see how many parts of planes you can find in the roofs of houses and other buildings. How many different planes enclose the whole house? Are any of the surfaces of the house curved, rather than plane?

In geometry we study the properties of space, the relationships between points, lines, planes, curves, surfaces, and solids, and the sizes and shapes of objects. Any fool can plainly see that we cannot develop space travel without knowing about space. Can you see it? Few people realize that we also use geometry to work out the best way for the Army to order shirts.

Imagine a navigator on a Coast Guard vessel which receives an S.O.S. from a sinking ship. From the information he receives from the radio operator he must be able to mark on his chart the position of the ship, and tell his captain in which direction he should steer. They must also know how fast they must travel in order to cover a certain distance before the ship sinks.

Some of the things you will study in geometry were already discovered by the Egyptians and Babylonians almost 4000 years ago. The Greeks, from about 600 B.C. to 200 B.C., made very important advances in geometry. The book of Euclid (about 300 B.C.) is still the basis of the geometry you will study in school. Archimedes, the greatest mathematician of antiquity, and others, such as Pythagoras, Eudoxus, Eratosthenes, and Apollonius, made very important contributions to geometry. After a long period during which very little new knowledge was added, geometry was revolutionized in the 17th century by Pascal, Descartes, and Fermat in France, who discovered how to apply algebra to geometry. The 19th century was a golden age of geometry in which completely new concepts of space were developed by Gauss and Riemann in Germany, Bolyai in Hungary, and Lobachevsky in Russia. About the beginning of the 20th century Poincare and Lebesgue in France and Hilbert in Germany introduced new ways of looking at the very fundamentals of geometry. Important discoveries are being made today by mathematicians in many parts of the world. We may mention, among others, Bing, Moise, Busemann, Milnor, and Nash in the United States, Pontryagin and the two Alexandrovs in Russia, Whitehead in England, Cartan, Leray, Serre, and Thom in France, Hopf and de Rham in Switzerland, Chern from China (now in U.S.), Yamabe from Japan (now in U.S.), and Papakyriakopoulos from Greece (now in U.S.).

There are still many important unsolved problems in geometry. For example, suppose you have a lot of marbles all exactly the same size. How should you pack them in a large container in the best possible way, that is, so as to get as many marbles as you can into the given volume? Nobody knows for sure. We know a very good way to pack the marbles, but no one has ever proved that it is the best possible.

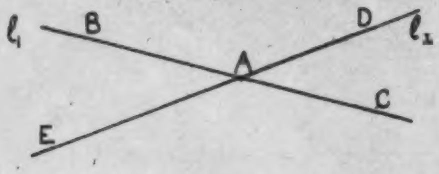
7. Find out where the word "geometry" comes from.
8. Look up one of the mathematicians mentioned above, say in an encyclopedia or *Bell*, *Men of Mathematics*, and write a short report on his life and at least one of his important discoveries.

Part A. SETS OF THREE LINES

When a mathematician begins an investigation, he usually starts with a very simple case. After he feels that he understands this case, he may then proceed to more complicated situations. In order to get a feeling for spatial relationships, let us begin by studying figures formed by three lines. In these two chapters, when we speak of a line, we always mean a complete straight line, extended as far as we please in both directions.

When we discuss geometrical objects such as points, lines, and angles, we find it convenient to give them names. Of course, we could call them "Joe" or "Wilhelmina", but we usually use just letters from the alphabet, and speak of the lines m and r, or the points A, P, and Y. Sometimes we use the same letter and attach different *subscripts*, small numerals written below and to the right, such as "l_1", "l_2", etc. The numerals here are simply labels to distinguish different lines or points, and the values of the numbers have nothing to do with the problems.

1. Draw two lines, and call them l_1 and l_2. Do l_1 and l_2 intersect in your drawing? If they do not, draw them again so they do intersect. Call their point of intersection point A. (Make the lines long enough to make it easy to measure the angles.)



2. How many angles are formed by l_1 and l_2?
3. Call the rays on l_1 AB ⟶ and AC ⟶. (Recall that AB ⟶ means the ray with A as endpoint and containing point B.) Call the rays on l_2 AD ⟶ and AE ⟶. Some of the pairs of angles which are formed have a common side. These are called adjacent angles. The other two pairs are called vertical angles.

   Each line divides the plane into two half-planes. One of each pair of vertical angles is the common part of two of these four half-planes, and the other vertical angle is the common part of the other two half-planes.
4. Measure the two angles of each pair of vertical angles with your protractor, and compare the measures. What is the relation between these measures?
5. Compare your results with those obtained by your classmates. What relation between the measures of vertical angles is true for each pair?

You have found by experiment, a certain relation to be true in a large number of cases. Let us see why this relation must be true in all cases.

6. a. What is the sum of the measures of angles DAB and BAE?

   b. What is the sum of the measures of the angles CAE and BAE?

c. If the measure of angle BAE is 130 degrees, then what is the measure of angle DAB? What is the measure of angle CAE?

d. Let x, y, and z be the measures of angles DAB, BAE, and CAE, respectively.

State your answers to parts (a) and (b) in the form of equations:

$x + y =$, $z + y =$

If y is known, how can you calculate x and z? Express your answer in the form of equations.

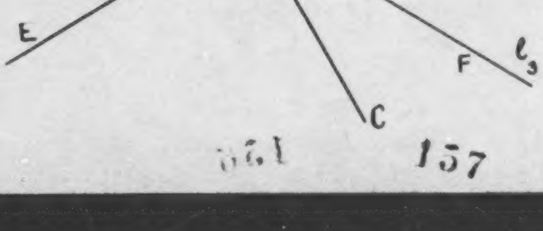
e. Does the relation which you found between x and z in part (d) agree with the result you obtained experimentally in exercise 5 above?

State the result in the form of a general principle by completing the following sentence:

Principle 1. When two lines intersect, angles in each pair of vertical angles formed

Compare your statement with those of your classmates. After discussion in class, try to write a clear and precise statement of the principle which all of you find satisfactory.

7. Draw another figure like the one you drew for exercise 1, and name the rays in the same way. Then draw another line on A and call it l_3. Call the other ray, on l_3 between AD → and AC →, AF → . Call the other ray, between AB →, AD → and AE →, AG →.



8. Start with angle BAD and name successive adjacent angles in clockwise order. How many of these angles are there altogether?

9. How many pairs of vertical angles can you find? Name each pair of vertical angles.

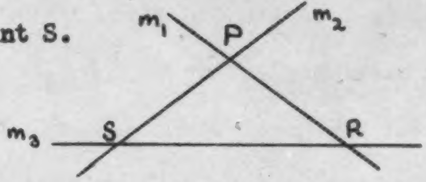
10. What is the sum of the measures of six angles formed by the rays on l_1, l_2 and l_3?

11. How many angles lie on one side of l_1? What is the sum of their measures?

12. Are the answers for exercise 11 true for l_2? For l_3?

13. Draw two lines, m_1 and m_2, which intersect. Call the point of intersection point P.

14. Draw a third line, m_3, which intersects m_1 and m_2, but not in the point P. Call the intersection of m_1 and m_3 point R, and the intersection of m_2 and m_3 point S.

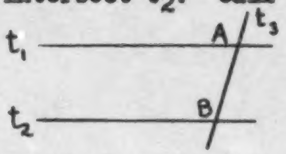


Ex 14 - 17

15. What is the name of the set of points made up of segments SP, PR, and RS?

16. How many angles in the figure have S as vertex? R? P? What kinds of angles pairs can you find?

17. Draw two lines t_1 and t_2 so they do not intersect. Draw a third line t_3 which intersects t_1 and call the point of intersection point A. Does t_3 intersect t_2? Call their point of intersection point B.



18. How many pairs of vertical angles are there in your figure? How many pairs of adjacent angles?

19. In what way are the last three figures you have drawn alike? In what way are they different? Can you think of a fourth way to draw a set of three lines? If so, draw three lines in this way.

20. The set of intersections of three lines in a plane may be _?_ points, _?_ points, _?_ points, or _?_ points.

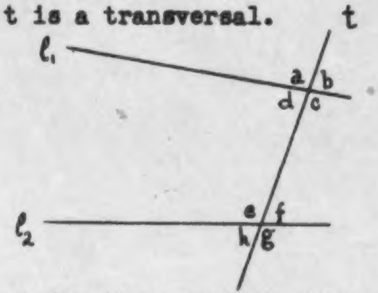
21. Look around you and find illustrations of sets of three lines like those in the figures you drew.

22. Imagine two lines in space. Is there any possible relation which could not occur in a plane?

23. What figures can be formed by the intersections of 3 planes in space?

Part B. TWO LINES AND A TRANSVERSAL

When a line intersects two other lines in distinct points, it is called a transversal of those lines. In the figure, t intersects lines l_1 and l_2, so line t is a transversal.



Ex. 3-8

1. In the figures you drew for the exercises in Part A, which ones show transversals?

2. In each figure in Part A which has a transversal, name the transversal and tell what lines it intersects.

3. In the figure above, how many angles are formed? How many pairs of vertical angles are there? What do you know about their measures? (You can say: "$\angle a \overset{m}{=} \angle c$". This means the measures of the angles are the same.)

4. What pairs of adjacent angles are there in the figure?

5. For what pairs of angles is the sum of their measures 180 degrees?

We give special names to some other kinds of angle-pairs in a figure like this. One kind is illustrated by angles b and f.

Notice that one of the rays which forms angle b is a part of a ray which forms angle f, and both angles are on the same side of the transversal t. Angles placed in this way are called corresponding angles.

6. What pairs of corresponding angles can you find in the above figure? There are four pairs in all.

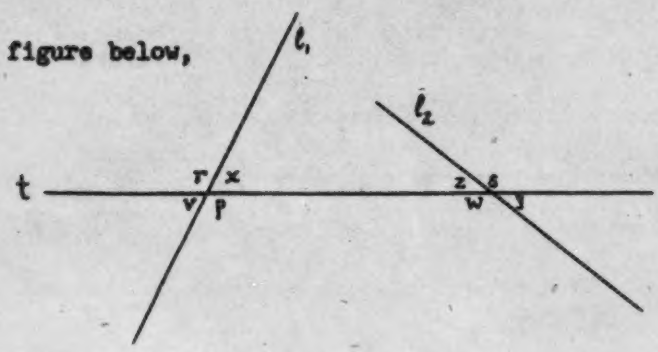
How can we distinguish the angles a, b, g, and h from angles c, d, e, and f? Angles a and b are on the opposite side of l_1 from the intersection of l_2 and t. They are called exterior angles. Similarly, angles g and h are exterior angles. Angles c and d are on the same side of l_1 as the intersection of l_2 and t. These angles and e and f are called interior angles. Angles d and f are interior angles with different vertices on opposite sides of the transversal t. They are called alternate interior angles.

7. Name the pairs of alternate interior angles in the figure above. There are only two pairs.

8. Name four pairs of angles in the above figure which have equal measures, no matter how the figure is drawn. State the reason why they are equal in measure.

Two angles whose measures add up to 180 degrees are called supplementary.

9. In the figure below,



name an angle which forms with angle x a pair of

- a. vertical angles
- b. corresponding angles
- c. alternate interior angles
- d. supplementary angles

10. In this figure name:

- a. four pairs of vertical angles
- b. four pairs of corresponding angles
- c. two pairs of alternate interior angles
- d. eight pairs of supplementary angles

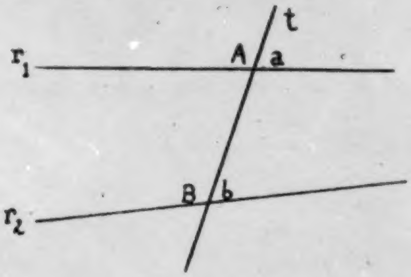
Notice how carefully we defined corresponding and alternate interior angles. Remember that we are discussing ideal points and lines, and that the streaks of printers ink are only meant to suggest a mental picture which you don't actually see. That is why we try to state the relations in terms which do not refer to the drawing. In mathematics we must say exactly what we mean, and must make our definitions as clear and precise as we can.

Part C. PARALLEL LINES AND CORRESPONDING ANGLES

Imagine that you are a civil engineer who must build roads intersecting a highway. In the figure below the highway is represented by the transversal t and the roads are represented by the lines r_1 and r_2.

You want to build the roads so that they don't intersect. Two lines in the same plane which do not intersect are called parallel. Railroad tracks form parallel lines.

You wish to tell your workers what angles the roads should make with the highway. Make a drawing with points A and B about $1\frac{1}{2}$ inches apart.



1. Make the measure of angle a 70 degrees and that of angle b 40 degrees. Do the roads intersect? If so, on which side of t?
2. Draw another figure with the measure of angle a 30 degrees and that of angle b still 40 degrees. What happens now?
3. Make at least ten experiments of this kind with various measures for the angles a and b. Record your results like this:

Measure of Angle a in degrees	Measure of Angle b in degrees	Intersection of r_1 and r_2
70	40	right of t
30	40	left
40	40	?

Can you find a general law which fits your observations? Compare your results with those of your classmates.

4. What kind of angle pair have you been measuring?
5. Make some more experiments to test the general law which you discovered in exercise 3. In each case first choose the measures of the angles, then predict whether r_1 and r_2 will intersect and if so, where, and finally make a drawing to check your prediction.

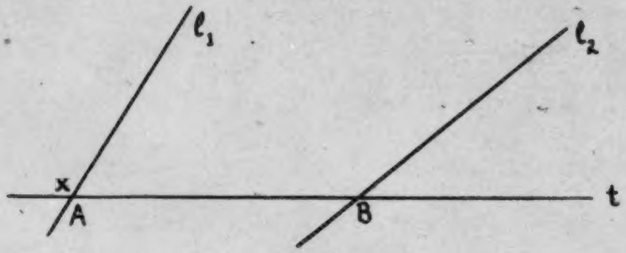
6. State your result by filling in the following sentence:

   Principle 2. When a transversal intersects two lines in the same plane, and

   then the lines are parallel.

   Discuss your statement with your fellow students. Try to arrive at a statement which is clear and precise and satisfies the class.

7. In the figure below, what angle forms with angle x a pair of corresponding angles? Label it angle p. If angles x and p have the following measures, do l_1 and l_2 intersect above or below t or are they parallel?



	Measure of Angle x in degrees	Measure of Angle p in degrees	Intersection of l_1 and l_2
a.	120	140	
b.	120	120	
c.	120	90	
d.	90	120	
e.	90	90	
f.	90	70	
g.	70	90	
h.	70	70	
i.	70	30	
j.	30	30	
k.	30	70	
l.	30	10	

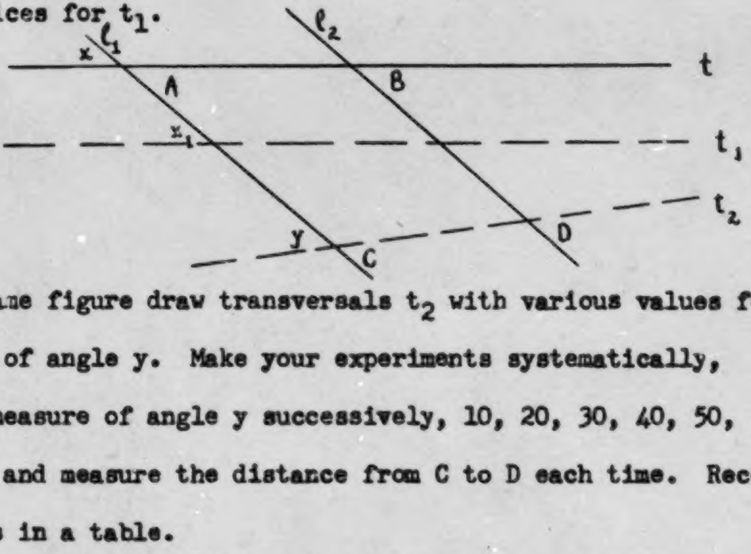
Check your predictions by drawing the figures.

State a general principle for deciding, on the basis of the measures a pair of corresponding angles, whether the lines will meet on the same side of the transversal as this pair of angles or on the other side.

Principle 3. When a transversal intersects two lines in the same plane and, of a pair of corresponding angles, the exterior angle has the larger measure, then

If, however, the interior angle of the pair has the larger measure, then

8. Draw the above figure so that angles x and p are 40 degrees in measure. Measure the distance from A to B. Now draw another transversal t_1 to l_1 and l_2 so that the measure of the angle x_1 formed by t_1 and l_1 is 40 degrees. What is the distance between the intersections of t_1 with l_1 and l_2? Try the experiment with several choices for t_1.



9. Using the same figure draw transversals t_2 with various values for the measure of angle y. Make your experiments systematically, making the measure of angle y successively, 10, 20, 30, 40, 50, ..., 170 degrees and measure the distance from C to D each time. Record your results in a table.

Measure of Angle y in degrees	Distance from C to D in inches
10	
20	
30	
.	
.	
.	
170	

Try various positions for the point C for each choice of the size of angle y. Does the distance from C to D depend on the position of C or on the angle y? For which value of the measure of angle y is the distance from C to D the least?

10. Try the same experiment with the distance from A to B twice as large as in the first case. Compare your two tables for the distance from C to D.

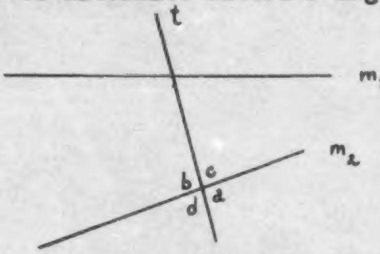
11. Draw a line parallel to a given line so that the minimum distance between the intersections of any transversal with the two lines is one inch.

12. Find in your classroom some parallel lines. Lay a yardstick or a ruler across them so as to form a transversal. Do it so that the intersections with the parallel lines are at a minimum distance apart. Measure this distance in each case.

Part D. PARALLEL LINES AND ALTERNATE INTERIOR ANGLES

You have just found a method for deciding whether two lines intersect by drawing a transversal to the lines and measuring a pair of corresponding angles. Let us see whether you can discover a criterion using alternate interior angles. We do this by applying a favorite device of the mathematician, namely by reducing the problem to the previous case.

1. In this figure, the transversal t intersects the lines m_1 and m_2. Which angle forms with b a pair of alternate interior angles? Label it angle e.



2. Suppose that the measure of angle b is 30 degrees. What are the measures of angles c, d, and a? Which of the previous three principles applies here?
3. Which of these forms with angle e a pair of corresponding angles?
4. If the measure of angle e is 20 degrees, will the lines m_1 and m_2 intersect, and if so, on which side of t? What about if angle e is 40 degrees in measure?
5. How can you tell, from measuring angles b and e, whether the lines m_1 and m_2 intersect, and if so, on which side? Apply the principles of the previous section.
6. State a general principle whereby you can say that m_1 and m_2 are parallel when you know the measures of a pair of alternate interior angles formed by the two lines and a transversal t.

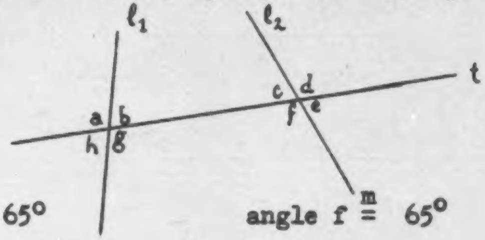
Principle 4.

Use principle 2 as a model in trying to write your own statement of principle 4. After comparing your statement with those of your classmates, discuss the matters one should think about when one is trying to state a mathematical proposition.

7. Now try to state a principle whereby you can tell on which side of the transversal the lines m_1 and m_2 will meet, from knowing the measures of a pair of alternate interior angles.

   Principle 5.

8. You can sometimes discover where improvements are needed in a statement when you try to apply it to problems. Predict in each of the following cases whether the lines l_1 and l_2 will intersect and if so, on which side of t, and tell which principles you used.

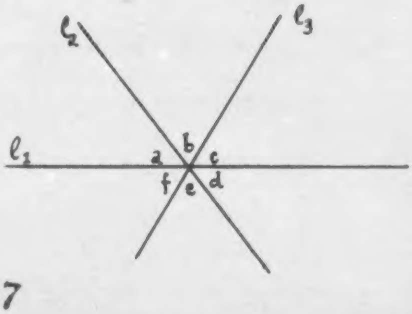


a. angle b $\overset{m}{=}$ 65° angle f $\overset{m}{=}$ 65°
b. angle g $\overset{m}{=}$ 60° angle e $\overset{m}{=}$ 60°
c. angle a $\overset{m}{=}$ 50° angle c $\overset{m}{=}$ 40°.
d. angle h $\overset{m}{=}$ 90° angle c $\overset{m}{=}$ 100°
e. angle g $\overset{m}{=}$ 130° angle d $\overset{m}{=}$ 50^c
f. angle b $\overset{m}{=}$ 50° angle c $\overset{m}{=}$ 120°
g. angle a $\overset{m}{=}$ 130° angle d $\overset{m}{=}$ 70°
h. angle h $\overset{m}{=}$ 40° angle d $\overset{m}{=}$ 70°
i. angle g $\overset{m}{=}$ 60° angle c $\overset{m}{=}$ 50°
j. angle b $\overset{m}{=}$ 80° angle d $\overset{m}{=}$ 80°

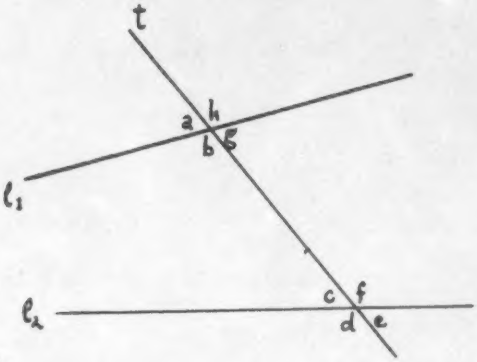
9. Draw figures for each of these cases and check your predictions.

Exercises

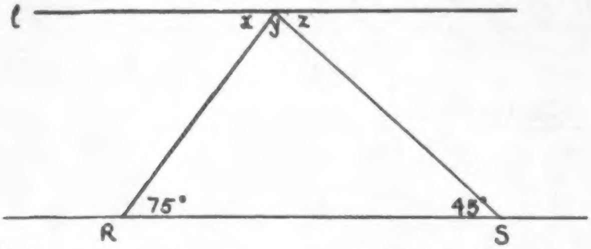
1. Make a list of the new kinds of angle pairs you have learned about.
2. In the figure, name the following:
   a. Two pairs of vertical angles.
   b. Two pairs of adjacent angles.
   c. A set of three angles the sum of whose measures is 180 degrees.



3. In the figure, name:
   - a. Two pairs of vertical angles.
   - b. Two pairs of adjacent angles.
   - c. A pair of angles whose sum is 180 degrees.
   - d. Two pairs of corresponding angles.
   - e. Two pairs of alternate interior angles.



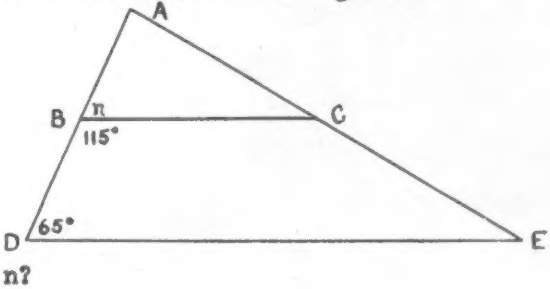
4. a. In this figure, draw the line 1 parallel to R S.



What must the measure of angle x be in order that 1 be parallel to R S? What principle did you use?

   - b. What is the measure of angle z? Why?
   - c. What is the sum of the measures of the angles x, y, and z?
   - d. What is the measure of angle y?

5. a. In the figure, what is the sum of the measures of angle CBD and angle BDE?



   - b. What is the measure of angle n?
   - c. Is BC parallel to DE? Why?

6. Cut a triangle out of cardboard, making the sides of different lengths. Then draw parallel lines by (1) placing one side of the triangle along the edge of your paper; (2) drawing a line along

another side of the triangle; then (3) sliding the triangle along the edge of the paper; and (4) drawing a line along the same side of the triangle.

b. What principle is illustrated?

Part E. TURNING A STATEMENT AROUND (Converse)

We nave seen that if certain things are true, then certain other things are true. For example,

1. *If* two angles are vertical angles, *then* the angles have the same measure. Suppose we make a new statement, by interchanging the "if" part and the "then" part. This is the new statement:

   If two angles have the same measure, *then* the angles are vertical angles. The new statement, obtained by interchanging the "if" clause with the "then" clause, is called a *converse* of the first statement.

   We have found that the first statement is true. Is the second statement always true? Must any two angles which have the same measure be vertical angles? If your answer is "No", then make a drawing of a case in which this statement is false.

Look at this statement.

2. *If* Mary and Sue are sisters, *then* Mary and Sue have the same parents.

   Is this statement true?

   Let us make the converse of this statement, by interchanging the "if" clause and the "then" clause.

   If Mary and Sue have the same parents, *then* Mary and Sue are sisters.

   Is this statement also true? Yes, it is true.

   We can see from these two illustrations that, if a statement is true, its converse, obtained by interchanging the "if" part and the "then" part, may be true or may be false.

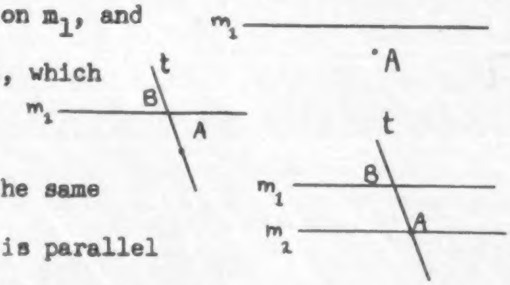
3. If you turn a true statement around, will the new statement also be true? Look at the statements below, and tell whether each one is true. Then write the converse, and tell whether the converse is necessarily true.
    a. If Rover is a dog, then Rover has four feet.
    b. If Blackie is a dog, then Blackie is a cocker spaniel.
    c. If a figure is a circle, then it is a closed curve.
    d. If the time is 10 A. M., Mary should be in school.
    e. If a figure is a closed curve composed of three line segments, then the figure is a triangle.
4. Make a table showing which of the above statements are true and which of the converse are true.
5. Make up five true statements, then state their converses. Make up some so that the new statement is true, and some for which it is false.

You can see from exercises 1-5 that when a true statement is turned around to form a converse, the converse is sometimes true and sometimes false.

Part F. CONVERSES OF PRINCIPLE 2 AND PRINCIPLE 4

Let us see whether the converse of Principle 2 seems to be true or false.

1. Draw a line m_1. Mark a point which is not on m_1, and call the point A. Through A, draw a line t, which intersects m_1 at point called B.
2. Using your protractor to make angles with the same measure, draw a line through point A which is parallel



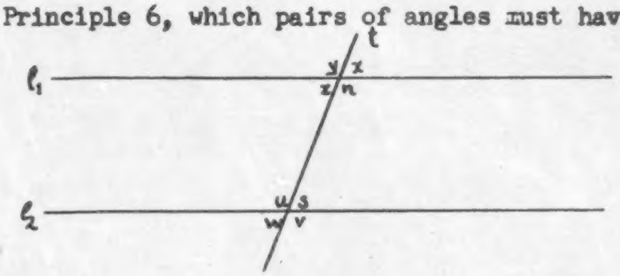
to line m_1. (What two kinds of angle pairs could you use?) Call this line m_2.

3. Now draw a second transversal intersecting m_1 and m_2. Call it t_1. Measure a pair of corresponding angles along transversal t_1. Are the measures of the corresponding angles the same?

4. Compare your results with those of your classmates. Do your angles have the same measures as theirs? What is true in each case? State your conclusion in the form of a general principle which tells the relation between corresponding angles formed when a transversal intersects two parallel lines.

   Principle 6.

5. In the figure below, suppose that line l_1 is parallel to line l_2. According to Principle 6, which pairs of angles must have the same measure?



Ex. 5-7

6. What other angle pairs will then have the same measure? Why?

7. If l_1 is parallel to l_2 then angle z $\overset{m}{=}$ angle w. Why?

   Angle w $\overset{m}{=}$ angle s. Why?

   Then angle z $\overset{m}{=}$ angle s. Why?

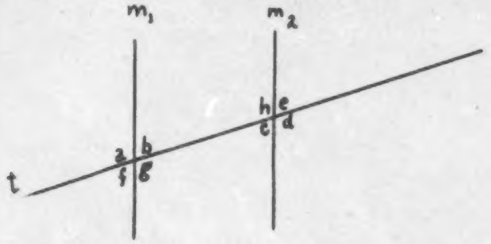
   What kind of angle pairs are angle z and angle s? Why?

8. Can you state a new principle from the reasoning you have done in Exercise 7?

   Principle 7.

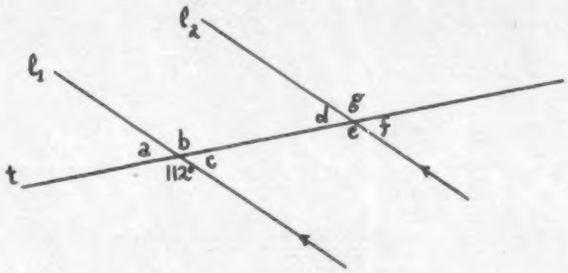
9. In the figure below, if line m_1 is parallel to line m_2 and a transversal t intersects them, which angle-pairs have the same measure because of Principle 7?

10. What other angle-pairs have the same measure? Why?

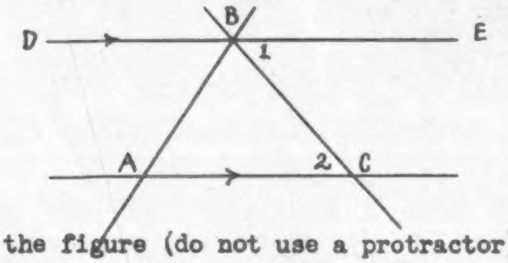


Ex 9 -10

11. In the figure, if l_1 is parallel to l_2 and the measure of one angle is 112 degrees, as shown, what is the measure of each of the other angles? (The arrows on l_1 and l_2 remind us that those are parallel lines.)



12. In this figure, ABC is a triangle, and DE is a line on point B. Also, DE is parallel to AC. How many transversals intersect DE and AC?

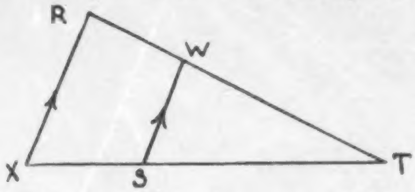


Ex. 12 - 14

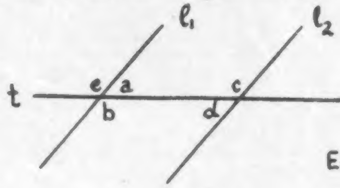
13. Make a sketch of the figure (do not use a protractor), but leave out line AB. What kind of angle pairs are angle 1 and angle 2?

14. Make another drawing of the figure, and this time leave out BC. What kinds of angle pairs do you see? Name the same angle pairs in the figure with all lines left in. (Use the three letter way of naming angles.)

15. In the figure below, suppose WS is parallel to RX. How many transversals intersect RX and WS? Find a pair of angles which have the same measure. Can you find a second pair?



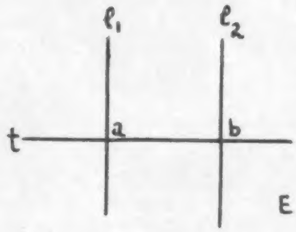
16 In this figure, l_1 is parallel to l_2 and t is a transversal. What is the sum of the measures of angles a and b? Why? What is true of angle b and angle c? Why? Then what is true of angle a and angle c? Would the same thing be true about angle b and angle d?



Ex. 16-17

17. Can you prove that angle a + angle c $\overset{m}{=}$ 180 degrees by using angle e instead of angle b? Explain your reasoning.

18. In this figure, if l_1 is parallel to l_2, and t is perpendicular to l_1, what else must be true of l_2 and t?



Ex. 18-19

19. Can you give reasons for these statements about the figure?

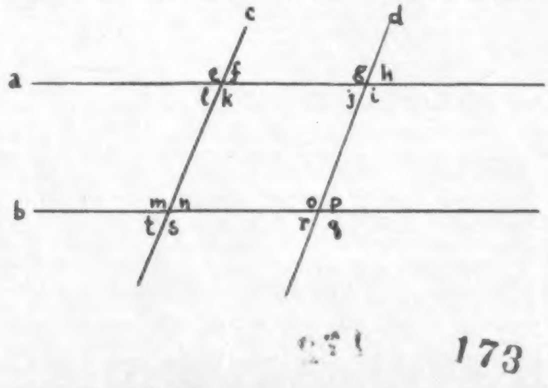
a. If t is perpendicular to l_1, angle a $\overset{m}{=}$ 90 degrees. Why?

b. If l_1 is parallel to l_2, angle b $\overset{m}{=}$ angle a. Why?

c. Then angle b $\overset{m}{=}$ 90 degrees. Why?

d. Then t is perpendicular to l_2. Why?

20. Here is a picture of two sets of railroad tracks which cross.



a. How many transversals do you see? Tell which lines each transversal intersects. Which lines are parallel?

b. Make a list of all pairs of corresponding angles.

c. List all pairs of alternate interior angles.

d. List all pairs of vertical angles.

e. List all pairs of non-adjacent angles whose measures have a sum of 180 degrees.

f. Suppose the measure of angle e is 152 degrees. Find the measures of as many other angles as you can.

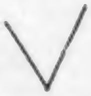
TRIANGLES

You have been discovering angle relations in a figure composed of three lines, two parallel lines and a transversal. Suppose the two lines are not parallel, so that each line is a transversal intersecting the other two lines. How does such a figure look?

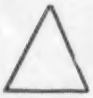
The three lines intersect in three points and, as you know, three points (not on the same line) and the segments joining them in pairs are a triangle.

1. Which of the figures below are triangles? If a figure is not a triangle, tell which requirement of the definition it lacks.

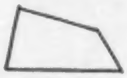
a



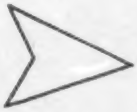
b



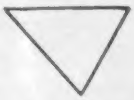
c.



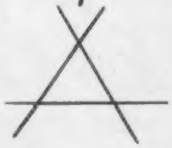
d



e



f

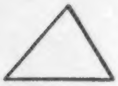


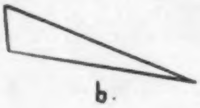
2. Draw a triangle which has two sides with the same measure. Such a triangle is called an isosceles triangle.

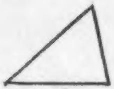
3. Draw a triangle which has three sides with the same measure. It is called an equilateral triangle.

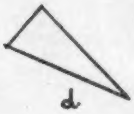
4. Draw a triangle with no two sides with the same measure. It is called a scalene triangle.

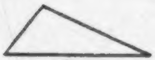
5. Which of the triangles below appear to be equilateral? Scalene? Isosceles?


a.


b.


c.


d.


e.

6. Draw two equilateral triangles, two isosceles triangles and two scalene triangles.

7. If a triangle is equilateral, is it also an isosceles triangle?

8. If possible, draw triangles whose sides have these measures: 3, 4, 5; 2, 7, 8; 1, 6, 7; 1, 6, 9. Which of these measures are not possible for the sides of a triangle? Why?

9. How can you tell whether a triangle can be formed if you know the measures of the three line segments?

If the distance from the point A to the point B is five inches, and the distance from B to C is two inches, what is the greatest possible measure for the distance from A to C? What is the least possible measure for this distance? Draw figures to illustrate both extreme cases and several intermediate cases.

If A, B, and C are the vertices of a triangle, then the angles ABC, BCA and CAB are called the angles of the triangle. The triangle is often called the triangle ABC.

10. Draw a large isosceles triangle ABC, making the sides AB and BC equal in length, but different from AC. Cut out the triangle carefully from paper, and then fold it so that AB lies exactly on CB. What do you notice about the angles CAB and ACB? Is one larger than the other or do they fit exactly, too? Try this experiment

with a number of isosceles triangles. Compare your results with those of your classmates.

State a general principle about the relation between the measures of the angles opposite two sides of a triangle which are equal in length.

Principle 8.

Do at least five experiments like this, constructing triangles with two angles equal in measure. Compare your results with those of your classmates.

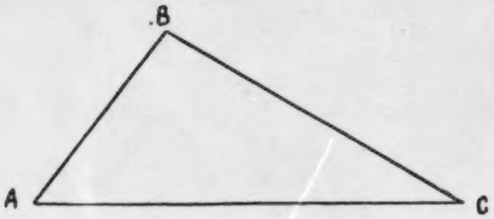
State a general principle about the relation between the lengths of the sides opposite two angles of a triangle which are equal in measure.

Principle 9.

11. Draw a line segment about 4 inches long, and at each end measure an angle with the same measure. About 40 degrees is a good size. Extend the sides of the two angles until they intersect to form a triangle. Cut out the triangle and fold it so that the one angle fits exactly over the angle which has the same measure. Do the sides fit exactly too?
12. Use the relation you observed in exercise 11 to show that if a triangle has three sides equal in length, then all three angles are equal in measure also.
13. Use the result of exercise 12 to show that if a triangle has three angles equal in measure, then it has three sides equal in length also.
14. Any two equilateral triangles have the same shape, that is, one may be thought of as a picture of the other, but on a smaller scale.

Is the same thing true of all isosceles triangles? Draw several isosceles triangles to illustrate your answer.

15. In exercises 10-11 you investigated the relations when the measures of two sides or two angles of a triangle are equal. Now you may explore these relations in scalene triangles. Make at least ten triangles with unequal sides and measure their angles. Record your results in a table like this:



BC	∠ CAB	AB	∠ BCA	AC	∠ ABC

Give the measures of the sides in inches and of the angles in degrees.

A good scientist arranges his observations systematically. You have here 6 quantities which may have various values. In order to discover the relations between them, it is best to hold several of them fixed, vary one of the others, and measure the rest. You will probably find it convenient to choose values for the lengths of BC and AB and keep these the same throughout your experiment. Then choose different values for the length of AC. Do the lengths of the sides determine the size and shape of a triangle completely? Then measure the angles, and record your observations as above. Your results in exercises 8-9 should give you information about the possible choices for the length of AC.

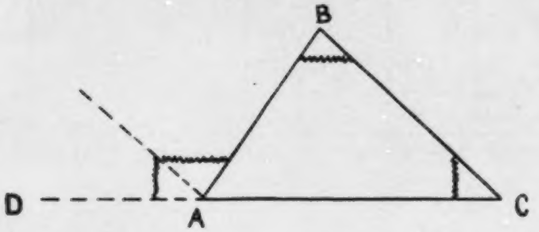
Notice that the column for each angle is next to that for the opposite side.

You and your classmates should each choose different lengths for BC and AB. In this way you will have plenty of experimental data for comparison.

On the basis of your observations you should arrive at certain conclusions. When one side is longer than another, how do the measures of the opposite angles compare? Is there a simple relation between the lengths of the sides and the measures of the angles? If you double the length of AC, does the measure of ∠ ABC change in a simple way? What happens if you double the lengths of all three sides of a triangle?

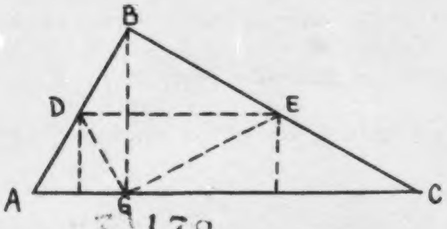
Part H. ANGLES OF A TRIANGLE

1. Draw a triangle ABC and cut it out of paper. Tear off the corners



at B and C and mount the whole figure on cardboard as shown, with the cut out corners pasted in around the vertex A. Measure the angle DAC. Compare the result of your experiment with those of your classmates. Is there a general law which fits your observations?

2. Cut a triangle ABC out of paper and mark off the midpoints D and E of the sides AB and BC respectively. (The midpoint is halfway

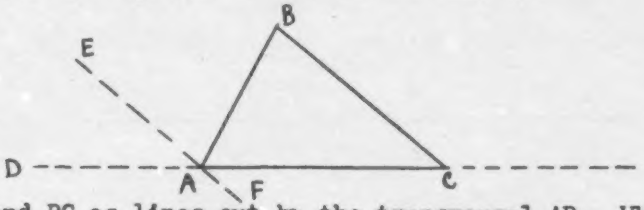


from one endpoint to the other, so that AD $\overset{m}{=}$ DB, BE $\overset{m}{=}$ EC.) Fold over the segment AD so that A lies over some point G of the line AC. Similarly, fold over EC so that C lies over G. Finally, fold along the segment DE. Where does the point B fall? Where do the segments DB and EB lie? What is the sum of the measures of the angles of the triangle? Do this experiment with other paper triangles. Does the law you found in exercise 1 fit these observations? State your law in the form of a principle concerning the sum of the measures of the angles of a triangle.

Principle 10.

You have now discovered a certain law by experiment. Were these experiments necessary? Is there any connection between this principle and the ones you found before? We shall now try to help you discover a proof of principle 10 from the previous ones. In other words, if you know the other principles, then you can arrive at this one by logical reasoning without a single additional experiment.

3. The experiment of exercise 1 suggests one line of reasoning. Instead of tearing off corners, simply draw a line AE so that Angle EAB has the same measure as angle ABC. Extend the line AC.



Think of AE and BC as lines cut by the transversal AB. What kind of angle pair are angles EAB and ABC? What conclusion can you draw from principle 4?

Now think of AE and BC as lines cut by the transversal AC. What kind of angle pairs are angles DAE and ACB? Does either principle 6 or 7 apply? What is the sum of the measures of the angles DAE, EAB, and BAC? What do you conclude about the sum of the measures of the triangle ABC?

4. Instead of working with the angle DAE, we could have compared angle ACB with angle FAC. Which principle applies in this case? Write out the complete proof from beginning to end, making any necessary changes, and giving the reason for each step of your proof.

   Compare your proof with those of your classmates. Did you omit any necessary steps? Did you have a convincing reason for every step? Was each statement in your proof clear and precise? Was your conclusion stated in such a way that everyone else understood exactly what you proved?

   Discuss with your classmates the things which make a proof good. Are there any common faults or mistakes which you must watch out for? After the class discussion rewrite your proof in accordance with the points brought out by you and your classmates.

5. In each of the following cases the measures of certain parts of the triangle ABC are given, those of the sides in inches and those of the angles in degrees, and you are asked to predict the measure of some other part. In each case, give your reason.

Given	To Find	Principle
a. $\angle ABC \overset{m}{=} 60^\circ$, $\angle BCA \overset{m}{=} 40^\circ$	$\angle CAB$	
b. $\angle ABC \overset{m}{=} 90^\circ$, $\angle CAB \overset{m}{=} 20^\circ$	$\angle BCA$	
c. $\angle CAB \overset{m}{=} 20^\circ$, $\angle BCA \overset{m}{=} 30^\circ$	$\angle ABC$	
d. $\angle BCA \overset{m}{=} 110^\circ$, $\angle ABC \overset{m}{=} 40^\circ$	$\angle CAB$	
e. $\angle ABC \overset{m}{=} 52^\circ$, $\angle BCA \overset{m}{=} 37^\circ$	$\angle CAB$	
f. $\angle ABC \overset{m}{=} 40^\circ$, $AB \overset{m}{=} 2$ in., $AC \overset{m}{=} 2$ in.	$\angle ACB$	
g. $AB \overset{m}{=} 3$ in., $BC \overset{m}{=} 3$ in., $CA \overset{m}{=} 3$ in.	$\angle BCA$	
h. $\angle BAC \overset{m}{=} 70^\circ$, $\angle BCA \overset{m}{=} 70^\circ$, $AB \overset{m}{=} 4$ in.	BC	
i. $\angle BAC \overset{m}{=} 100^\circ$, $\angle BCA \overset{m}{=} 40^\circ$, $AB \overset{m}{=} 4$ in.	AC	
j. $\angle ABC \overset{m}{=} 60^\circ$, $\angle CAB \overset{m}{=} 60^\circ$, $AC \overset{m}{=} 3$ in.	AB	

Check your answers by drawing figures and measuring.

6. Suppose one angle of an isosceles triangle has a measure of 50°. Find the measures of the other two angles of the triangle. Are two different answers possible?

7. In triangles ABC and DEF, suppose that $\angle BAC \overset{m}{=} \angle EDF$ and $\angle BCA \overset{m}{=} \angle EFD$. What will be true about angles ABC and DEF? Why?

Informal Geometry I

Statement of Principles

Principle 1. When two lines intersect, the angles in each pair of vertical angles formed have equal measures.

Principle 2. When a transversal intersects two lines in the same plane, and a pair of corresponding angles have the same measure, then the lines are parallel.

Principle 3. When a transversal intersects two lines in the same plane and, of a pair of corresponding angles, the exterior angle has the larger measure, then the lines intersect on the same side of the transversal as these two angles. If, however, the interior angle of the pair has the larger measure, then the lines meet on the opposite side of the transversal from this pair of angles.

Principle 4. When a transversal intersects two lines in the same plane, and a pair of alternate interior angles have the same measure, then the lines are parallel.

Principle 5. When a transversal intersects two lines in the same plane, and one of a pair of alternate interior angles is smaller than the other, then the lines intersect on the same side of the transversal as the smaller angle.

Principle 6. When a transversal intersects two parallel lines, then each pair of alternate interior angles have the same measure.

Principle 7. When a transversal intersects a pair of parallel lines, then each pair of alternate interior angles have the same measure.

Principle 8. If two sides of a triangle are equal in length, then the opposite angles are equal in measure.

Principle 9. If two angles of a triangle are equal in measure, then the opposite sides are equal in length.

Principle 10. In any triangle the sum of the measures of the angles is 180 degrees.

UNIT IX

INFORMAL GEOMETRY II

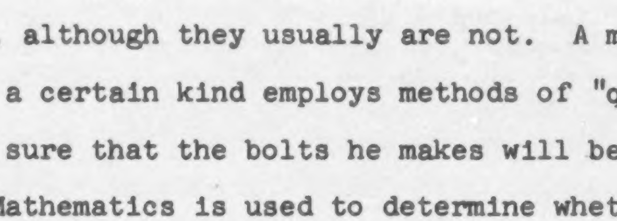
(Congruent triangles, Perpendicular bisectors, Parallelograms, Concurrent lines, and the Right Triangle Principle)

CONGRUENT TRIANGLES

When leaves are drifting to the ground in the fall, have you ever tried to find two leaves from the same tree which have exactly the same size and shape? While we can tell whether a leaf is from an oak tree or a maple tree or some other kind of a tree, it is unusual to find two oak leaves which seem to be exactly the same size and shape if we place one on top of the other to compare them. You may also have wondered why, among the faces of all the people whom you know, or even among all the people in a great city, you so seldom find two which are very much alike.

The wheat fields of two farmers may be "exactly" the same shape and size, although they usually are not. A manufacturer of large bolts of a certain kind employs methods of "quality control" to try to make sure that the bolts he makes will be as nearly alike as possible. Mathematics is used to determine whether objects that appear to be alike actually are, and to determine the amounts and importance of difference in the objects.

In order to study the characteristics of objects that are exactly alike, we begin, as scientists usually do, with a very simple situation. It turns out that the case of two triangles is the key to the general problem of determining when two objects have exactly the same size and shape.



These triangles appear to be the same size and shape; if we cut one of the triangles out of the paper, and find that we can place it on the other so that the three vertices and sides of one triangle lie exactly on the three vertices and sides of the other triangle, we say that the two triangles are congruent. Or, if we could move one of the triangles in its plane until it fits exactly on the other, we would say that the triangles are congruent.

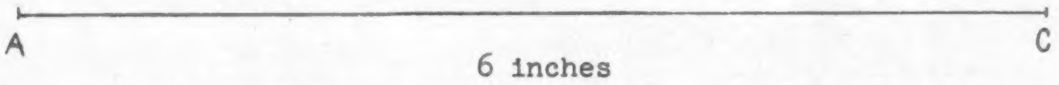
Principle 1: Corresponding segments and angles of congruent figures have equal measures.

If one triangle were cut out of a rubber sheet, then we could stretch or shrink parts of it while we move it, and thus make it fit on the other. In such a case we would certainly not call the triangles congruent. It is essential that the motion of the triangle be rigid, that no distances between points of the figure should change during the motion.

We could compare solid figures in a similar way by making a mold to fit one solid, and then testing the other solid by trying the mold on it. Again, in this process we must be sure that, when we move things around, the motions are rigid.

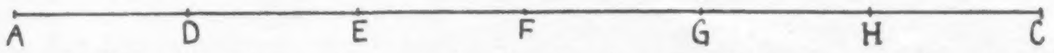
Now let us look more carefully at this method of comparing figures, and see what it really means. If we understand how it works, we may be able to discover a method of comparison which does not require a physical motion. When we fit one figure onto another, we make each point of the first figure correspond to a certain point of the other point for point. The distance between any two points of the first figure is the same as the distance between corresponding points of the other. Since we bring the first figure to fit on the other by a rigid motion, during which no distance is changed, then the points of the original figures can be matched with each other so that the distance between any pair of points in the first figure is the same as the distance between the corresponding pair of points in the other.

For example, draw triangles A B C and A' B' C' so that the lengths of A B and A' B' are 4 inches, those of B C and B' C' are 5 inches, and those of C A and C' A' are 6 inches. (How can you draw a triangle with sides of given length? Suppose you locate A and C 6 inches apart. Where should B be?



Forget about the length of B C for a moment and concentrate on the requirement that A B should be 4 inches long. Where do the points lie whose distance from A is 4 inches? Do you have any instrument for drawing the figure formed by these points? Next draw the curve formed by all points whose distance from C is 5 inches. Where are the possible positions for B? How many are there?) Cut out triangle A B C and see if this triangle is congruent to A' B' C' by fitting it onto the latter.

Now separate the triangles again. Using your ruler, mark off on the side AC points D, E, F, G, and H, so that



the distances from A are 1, 2, 3, 4, and 5 inches respectively, and mark off on the side A B points J, K, and L whose distances from A are 1, 2, and 3 inches respectively. Can you predict the positions of these points when you place the triangle A B C on A' B' C' so that A lies on A', B on B' and C on C'? Mark off your predictions on the triangle A' B' C' and label them D' E' F', . . ., L'. Now check your predictions.

Measure the distance between the various pairs in the first figure and record them in a table:

	A	B	C	D		L
A	0	5	6			
B		0	4			
C			0			
D						
L						

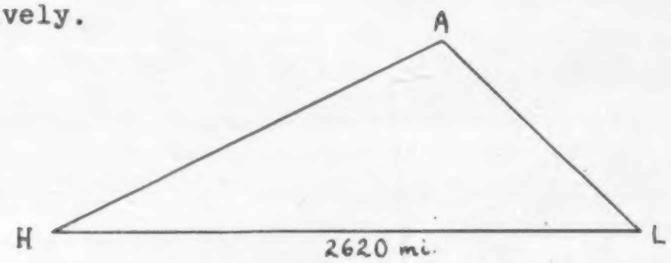
Make a corresponding table for the points A', B', . . ., L'. Compare distances between corresponding points.

Pick a point M in the interior of the triangle A B C. How can you predict the position of the corresponding point M' in the other triangle? Are there any lines which you can draw and distances which you can measure so that you could locate M' exactly? Make your prediction and check it by placing the triangle A B C upon the triangle A' B' C'. Does M lie above your prediction?

If we know the distance between every pair of points in a figure, this gives us complete information about its size and shape. In the above example of the triangle A B C, you can find as many points as you please on the three sides. You could measure all distances and give a complete description of the triangle. Is all this information necessary? If you know the lengths of the sides, and you locate the point D on A C by marking off the length of A D as 1 inch, does this information determine the distance from B to D?

Is more than one value for this distance possible? Incidentally, if D is on the side AC, and AD = 1, AC = 6, then what is the length of DC? How can you tell from the lengths of AD, DC, and AC whether D is on the segment AC? Then is there more than one possibility for the length of BD?

Sometimes you may know the sizes of certain angles and certain segments. Imagine that you are at point A in the figure below, and you are the navigator of an airplane. You receive signals from the radar operators at the airports in Los Angeles and Honolulu which tell you that the measures of the angles AHL and ALH are 20° and 30° respectively.



Your chart shows that the distance from Honolulu to Los Angeles is 2620 miles. Is this enough information for you to be able to determine your distance from Honolulu? Make a drawing to scale using 1/8 of an inch to represent 100 miles. How long should HL be? Is there more than one possible length for AH? Measure AH, and calculate the distance from your airplane to Honolulu.

The main problem of this unit is to find out what parts of a figure must be known in order to determine all the other parts. We could also think of the problem as that of investigating what parts of two figures should be measured in order to know whether they are congruent. This would enable us to find out whether figures are congruent without actually moving them. We have already obtained some evidence that it is not necessary to measure all parts of a triangle in order to determine the complete size and shape.

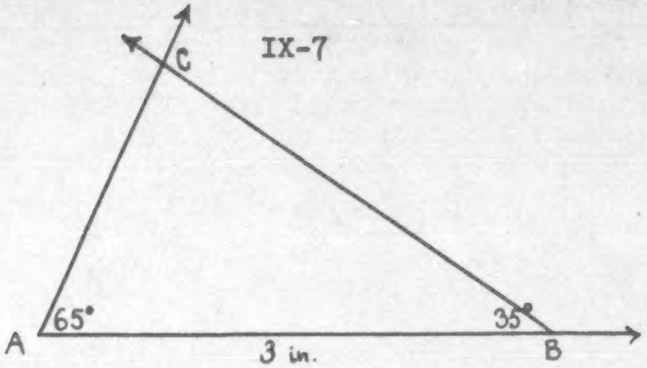
Before we tackle the general problem of when two figures are congruent, we start, like good mathematicians, with the simplest case, namely that of two triangles. After we thoroughly understand this case, we will be prepared to study more complicated figures.

A triangle has six parts -- three sides and three angles. We shall investigate which combinations of these parts are enough to determine the rest, and so to determine the size and shape of the the triangle.

Choose a point A and a ray with A as end point. Find a point B on the ray whose distance from A is 3 inches. How many possible positions are there for the point B? Choose one of the sides of the line AB and, using a protractor, draw a ray on this side with A as endpoint which makes with AB an angle whose measure is 65°. On this same side of AB draw another ray with B as endpoint which makes with AB an angle whose measure is 35°. Do these rays intersect? If so, label the intersection C. Once you have chosen the point A and drawn the ray on which B should lie, is there more than one possible way of drawing the triangle ABC according to the above directions?

Try this experiment again on another sheet of paper. Is the second triangle congruent to the first? Try this for three separate triangles, each time making the line segment equal in length to AB and drawing the angles with the measures and positions shown in the figure. Cut out two of the triangles and place them on the third one. What do you conclude?

In the figure, AB is the side included between the angle whose measures are 35 degrees and 65 degrees. To be more brief, we say, "two angles whose measures are 35 degrees and 65 degrees and the included side."



EXERCISES

1. Draw two triangles making one side of each triangle 3 inches long, and the angles on the sides with measures 40 degrees and 80 degrees. Measure the other two sides of each triangle and compare the lengths. What do you conclude?

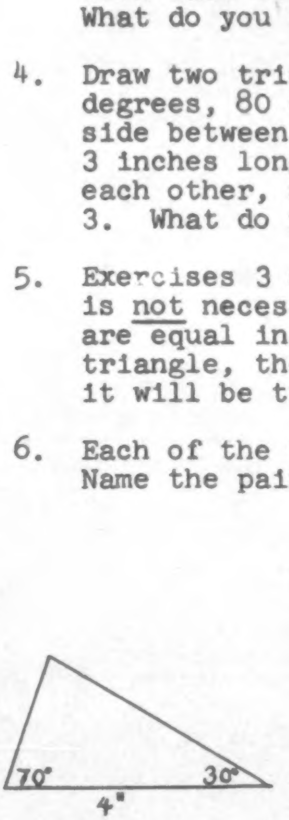
2. If two angles of a triangle measure 40 degrees and 80 degrees, what is the measure of the third angle?

3. Draw two triangles each of which has a side 3 inches long, and angles which measures 40 degrees, 80 degrees, and 60 degrees. Draw the 3 inch side between the vertices of the 40 degree and 60 degree angles. Cut out these triangles and compare them with each other, and with the triangles you drew in exercise 1. What do you find?

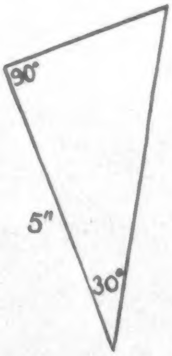
4. Draw two triangles, each of which has angles which measure 40 degrees, 80 degrees, and 60 degrees. In your drawing make the side between the vertices of the 60 degree and 80 degree angles 3 inches long. Cut out the triangles and compare them with each other, and with the triangles you drew for exercises 1 and 3. What do you find?

5. Exercises 3 and 4 should convince us that the following statement is not necessarily true: If two angles and one side of a triangle are equal in measure to two angles and one side of another triangle, the triangles are congruent. Correct the statement so it will be true.

6. Each of the triangles below is congruent to one of the others. Name the pairs of congruent triangles.



(a)



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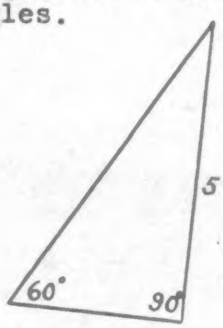

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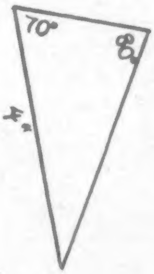
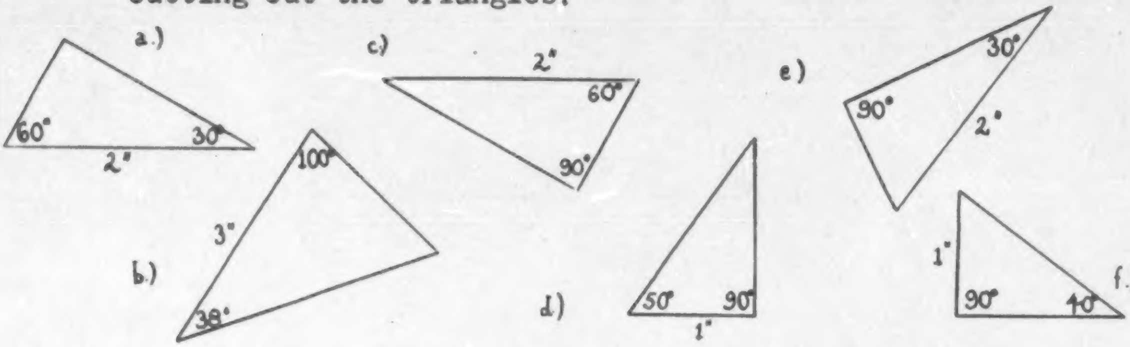
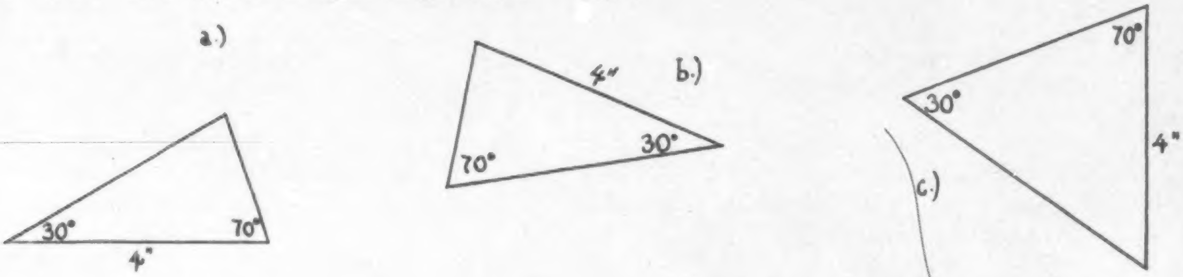


(d)

7. Group the following triangles into sets of congruent triangles. Can you do this without measuring any of the sides or angles, or cutting out the triangles?



8. Are the following congruent? Why?



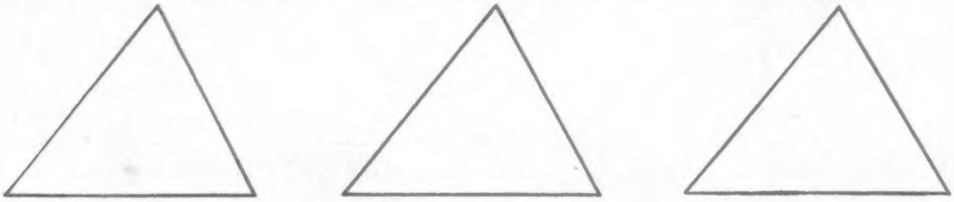
9. Draw a triangle with angles whose measures are 50°, 60°, and 70°. Can you draw a larger triangle whose angles are equal in measure to those of another triangle? Are the triangles necessarily congruent?

10. State the principle which you have found from the above exercises, whereby you can tell whether two triangles are congruent by measuring in each triangle two angles and the included side.

Principle 2:

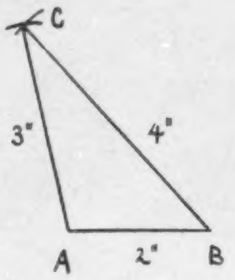
EXERCISES

1. Compare with a compass the sides of the three triangles below:



What do you conclude? Do the three triangles appear to be congruent?

2. Draw a triangle with sides whose lengths are 2 inches, 3 inches, and 4 inches. This is rather hard to do with a ruler, but easy if you use a compass. Follow these directions: Draw a line segment 2 inches long. Call it AB. Spread your compass so the points are three inches apart. With the point of the compass on point A, draw part of a circle above AB. Spread the compass so the points are 4 inches apart. Place the point of the compass on point B, and draw part of a circle so it intersects the part of a circle you drew before. Call the point of intersection point C. Draw line segments AC and BC. Are the sides of the triangle ABC 2 inches, 3 inches, and 4 inches long?



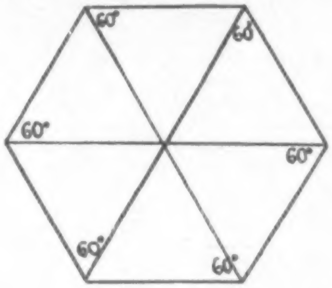
3. Draw a second triangle with sides 2 inches, 3 inches, and 4 inches long. Cut out the second triangle and test it to see whether it is congruent to the triangle you drew for exercise 2. What is your conclusion?

4. Draw three triangles with sides 2 inches, 2.5 inches, and 4 inches long. Test them to see whether they are congruent as you did before. What is your conclusion?

5. State a general principle whereby you can tell whether two triangles are congruent from measurements of their sides.

Principle 3:

6. Do you find any congruent triangles in this figure? On what do you base your answer?



7. Devise a method for constructing a triangle if you are told the measure of its three sides, using the following tools: A ruler, 3 thumb tacks, and a piece of string.

Principle 2 and principle 3 state two cases in which two triangles are congruent if certain three parts of one triangle are equal in measure to similarly placed parts in another triangle. A complete listing of sets of three parts of a triangle is as follows:

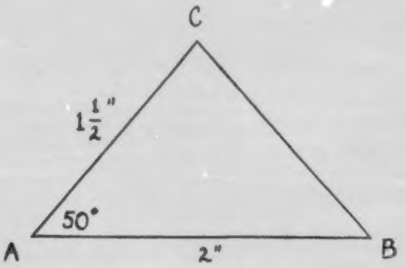
I. 3 angles
II. 3 sides
III. 2 angles and the included side
IV. 2 sides and the included angle
V. 2 angles and a side not included between the vertices of the angles
VI. 2 sides and an angle not included between the sides.

In principles 2 and 3 you have stated methods for deciding whether two triangles are congruent on the basis of measurements of the parts listed in III and II above. In exercise 9 on page 8 you investigated whether measurements of the parts mentioned in I are enough to enable you to tell whether two triangles anr congruent. What was your conclusion?

Recall principle 10 of the previous chapter. Is it necessary to measure all the three angles of a triangle in order to know their sizes? Can you apply this idea to case V above? Try to use the mathematician's favorite trick of reducing a problem to the previous case.

EXERCISES.

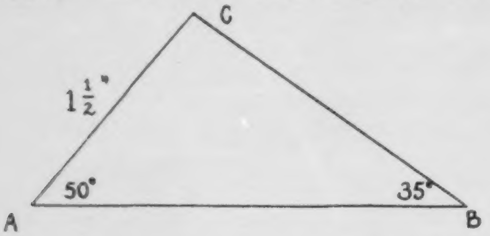
1. Consider the set of parts listed as IV, two sides and the included angle. The positions of the parts in this set are illustrated in the drawing below. Use the measures given and draw three triangles. Then test the triangles to see whether they are congruent.

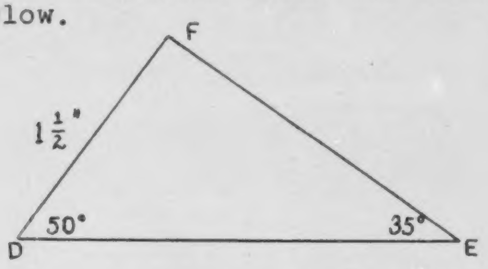


If the point A, the ray AB, and the side of AB on which C lies are given, is it possible to draw more than one triangle with sides and angles having the above measures?

2. Consider the set of parts listed as V, two angles and a side which is not included between the vertices of the angles.

   If two angles of one triangle have the same measures as two angles of another triangle, what will be true about the third angle? Look at the triangles below.





3. Now consider Case VI. Draw a triangle ABC so that AB = 3 inches, angle A = 30°, and BC = 2 inches. Let us see, how many possible shapes can such a triangle have? After you locate the points A and B 3 inches apart, then you still have to find the point C. Choose the side of the line AB where you want the triangle to lie. Where do all the points C lie whose distance is 2 inches from B? Draw the figure formed by this set of points. Now draw angle A making it 30°. How many positions are possible for C in this halfplane? What are the possible lengths for the side AC?

   Can you decide whether two triangles are congruent by measuring the parts in case VI?

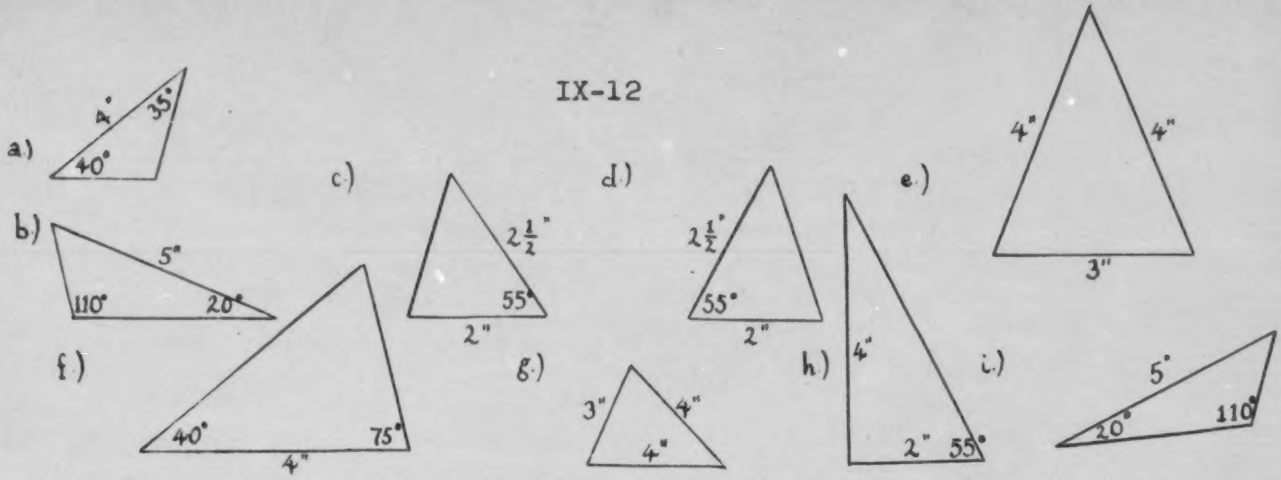
4. On the basis of your experiments in exercise 1, state a principle whereby you can decide whether two triangles are congruent by measuring the parts given in case IV.

Principle 4:

5. State your conclusions about the set of parts listed in V and the set of parts listed in VI.

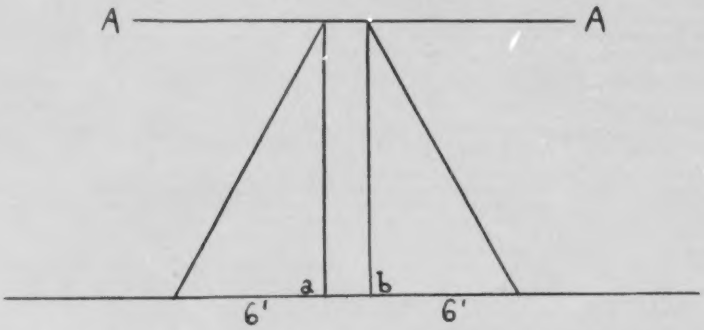
EXERCISES.

1. For each of the triangles below, describe the set of parts whose measures are marked. You may use the descriptions listed on page 10.



2. Name any two of the triangles above which are congruent, and tell which principle applies.

PERPENDICULAR BISECTORS



1. The telephone pole in the figure is supported by two wires, each of which is fastened to the ground at a point 6 feet from the base of the pole. The Telephone Company knows that these guy wires will be about equal in length. How do they know this?

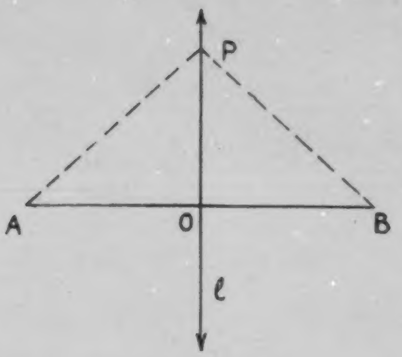
   A. The company has set up the pole so it is perpendicular to the ground. How large are angles A and B?

   B. In the two triangles the lines represented by the sides of the pole are equal in length.

   C. The sides of the triangles along the line representing the ground are each 6 feet long.

   What can you conclude from statements A, B, and C? Which principle applies?

2. In exercise 1, suppose the company attaches two more guy wires to the pole in the same way as in exercise 1. What will be true about the lengths of these guy wires? Why?



In the figure the line 1 is perpendicular to segment AB. The intersection of the sets of points, AB and 1, is the point O, and the measures of AO and OB are the same. So line 1 is called the perpendicular bisector of AB. Where does the word bisector come from?

We can show that if P is any element of the set 1 then the length of PA is equal to the length of PB. In order to show this consider the triangle POA and triangle POB.

$$AO \overset{m}{=} OB$$

$$\text{Angle } AOP \overset{m}{=} 90 \text{ degrees}$$

$$\text{Angle } BOP \overset{m}{=} 90 \text{ degrees}$$

$$OP = OP$$

Therefore triangle AOP is congruent to triangle BOP, and PA $\overset{m}{=}$ PB.

EXERCISES

1. a. In the argument above, why does AO $\overset{m}{=}$ OB?
   b. Why does angle AOP $\overset{m}{=}$ angle BOP?
   c. Why is triangle AOP congruent to triangle BOP?
   d. Why, then, is PA $\overset{m}{=}$ PB?

2. Select a different element of the set 1 and call it P. Would the argument above be changed in any way?

3. Consider the half-planes determined by the line on segment AB. If P is located on 1 in the other half-plane than the one shown in the figure will you need to change the argument in any way? Make a drawing to illustrate your answers.

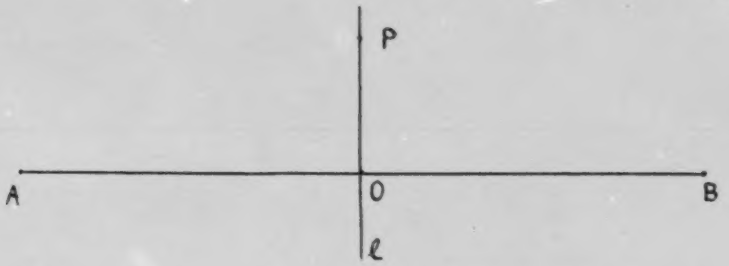
4. Draw a line segment. With ruler and protractor draw the perpendicular bisector of the segment.

5. a. Draw a line segment, and call it PR.
   b. Use ruler and protractor to draw the perpendicular bisector of PR. Call the perpendicular bisector 1 and its intersection with PR point A.

   c. Select an element of 1, and call it B.

5. d. Measure the distance from B to P and from B to R.
   e. What do you find is true? Is your conclusion the same as before?

6. Mark two points A and B on a sheet of tracing paper and draw the segment AB with a ruler. Fold over the paper so that B falls on A and the segment AB is folded over itself, and flatten the paper so that a crease is formed. Open it up and draw l a line l along the crease. Call the intersection of l with AB the point O, and choose any other point P on l.

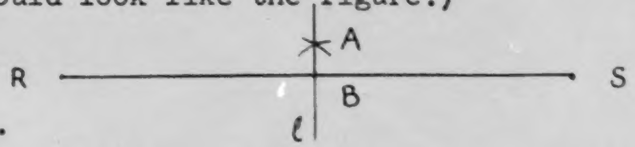


Since, when the paper is folded, the segment BO lies over the segment AO, what is true about the measures of these segments? Since the angle BOP lies over the angle AOP, what do you know about the measures of these angles? What is the sum of their measures? Can you tell what the measures of these angles must be without measuring them? Draw the segments PA and PB. If you fold the paper along l, where will PB lie? How are the lengths of PA and PB related?

7. State a principle about the distances from any point on the perpendicular bisector of a line segment to the endpoints of the segment.

Principle 5:

8. Draw a line segment, and call it RS. Then use a compass to locate a point, not on RS, which is the same distance from points R and S.
(Your drawing should look like the figure.)



Call the point A.

   b. Use your ruler to find the midpoint of RS. Call the midpoint B. Then draw AB.
   c. Does segment AB seem to be perpendicular to segment RS?
   d. Draw the line on AB and call it l. Draw segments AR and AS.

9. In your figure for exercise 1,

AR $\overset{m}{=}$ AS. Why?

BR $\overset{m}{=}$ BS. Why?

AB $\overset{m}{=}$ AB. Why?

So triangle ABR is congruent to triangle ABS. Why? Then how large are angle ABR and angle ABS? What is the sum of these angles? Does this show that l is the perpendicular bisector of RS?

10. Repeat exercise 8 locating a point A in the other half-plane determined by the line of RS. Does this change the argument in any way?

11. State a principle whereby you can tell whether a point lies on the perpendicular bisector of a line segment by measuring its distances from the endpoints.

Principle 6:

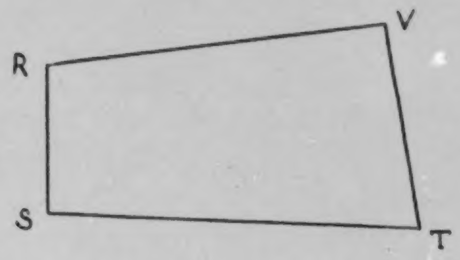
12. Compare the statements in exercises 7 and 11 above. What is the difference between them? Is one a converse of the other?

    We can use principle 6 to guide us in construction of a perpendicular bisector with ruler and compass.

12. a. Draw a line segment, and call it DC.
    b. Use compasses to locate a point E the same distance from C and D.
    c. Use compasses to locate a second point F the same distance from C and D, in the other half-plane determined by the line on CD.
    d. Draw the line on E and F. Call it l. Must E lie on the perpendicular bisector of CD? Must F lie on the perpendicular bisector of CD? Why? Is there any line on E and F, except l? Then is l the perpendicular bisector of CD?

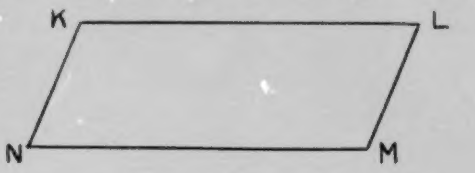
PARALLELOGRAMS

We call a figure with 3 points joined by 3 line segments a triangle. (Do you know what the prefix "tri" means? Think of the words like "triple", "trio", "tricycle", etc. Does this suggest the meaning of the word triangle?) We call a figure with 4 sides a quadrilateral. If R,S,T, and V are distinct points, then the figure formed by the segments RS,ST,TV, and VR is a quadrilateral, and is often called the quadrilateral RSTV. These 4 segments are its sides, and R, S, T, and V are called its vertices. The angles VRS, RST, STV, and TVR (also called "angles R, S, T, and V") are the angles of the quadrilateral.

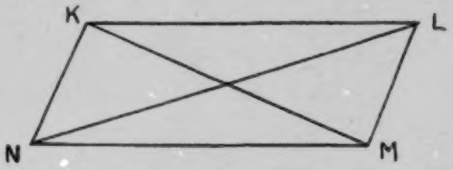


Since we have just made a rather thorough study of triangles, we are ready to attack problems about more complicated figures, and the next step is, of course, the study of quadrilaterals. Certain special kinds of quadrilaterals are important enough to be worth more detailed study.

A quadrilateral in which the pairs of opposite sides are parallel is called a parallelogram. This figure KLMN is a parallelogram.



The line segments, KM and LM, are called the diagonals of the parallelogram. The diagonals shown in the next figure.



By using the congruent triangles we can rather easily show that

a. If a quadrilateral is a parallelogram, then the opposite sides have equal measures.

b. If a quadrilateral is a parallelogram, the diagonals bisect each other.

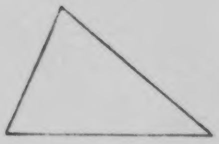
You could draw parallelograms and measure their sides and diagonals and hence conclude by experiment that these statements are probably true.

In the experimental method, however, we shall always have some uncertainty because the measurements are never exact, only approximate, and because we can at most test these statements in a large number of cases, so that we could not be sure that they are always true. Let us try to apply the deductive method instead.

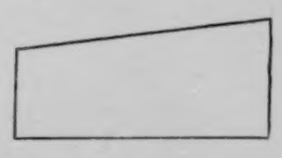
EXERCISES

1. Which of these figures are quadrilaterals?

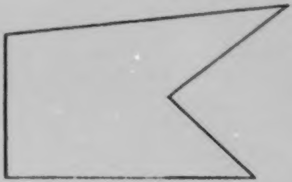
a)



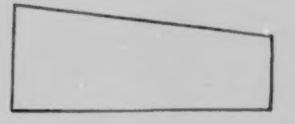
b)



c)



d)

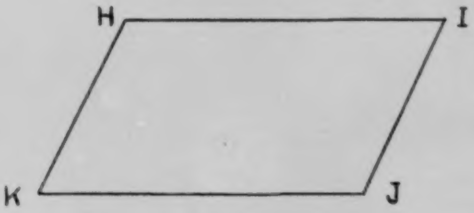


2. Which of the figures in exercise 1 appear to be parallelograms?

3. Draw two parallelograms of different sizes and by measurement, see if the statement (a) page 16 appears to be true.

4. Construct a parallelogram ABCD and cut it out. Cut the parallelogram along the diagonal AC. Are the triangles you obtained congruent? Check by fitting one onto the other. What sides and angles of the parallelogram have equal measures?

5. Let us try to help you discover for yourself why statement (a) is true. If you answer the following questions, referring to the parallelogram below, you may



be able to deduce the statement from the principles you already know.

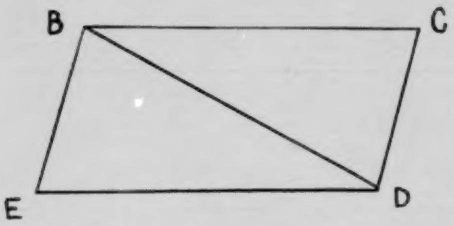
a. Which principles state that under certain circumstances two segments are equal in length?

b. Do any of them apply immediately to the above figure? If not, can you draw a segment in such a way that one of these principles might apply? Think of the preceding exercise.

c. What kind of lines are HI and JK? HK and IJ? How is the line you have just drawn related to these lines? Are there any angles which must have equal measures because of this?

d. Can you apply any of the principles of this chapter in order to show that certain triangles are congruent?

6. The figure BCDE below is a parallelogram.



In the following reasoning, tell why the statement in each step is true.

a. BC is parallel to DE

b. $\angle$ CBD $\overset{m}{=}$ $\angle$ BDE

c. EB is parallel to CD

d. $\angle$ CDB $\overset{m}{=}$ $\angle$ EBD

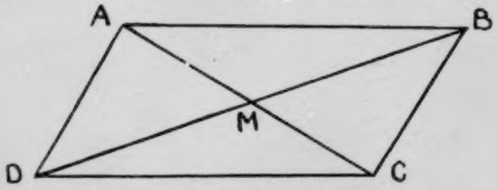
e. BD $\overset{m}{=}$ BD.

f. The triangle BCD is congruent to the triangle DEB.

g. BC $\overset{m}{=}$ DE.

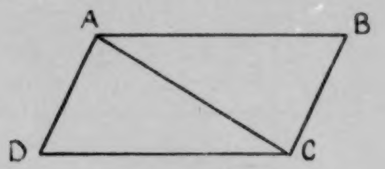
7. Draw two parallelograms of different sizes and, by making measurements, see if the statement b on page 16 appears to be true.

8. Show why the statement b. on page 16 is true. Give your reasons carefully. In the figure you might show that triangles AMB and CMD are congruent. If you think of alternate interior angles you can find angles with equal measures. Also statement a on page 16 may assist you in supporting this statement.



9. Draw a quadrilateral ABCD in which AB = DC and BC = AD. Measure the angles and apply one of the principles of the previous chapter. How are the opposite sides related? What special kind of quadrilateral is ABCD? Repeat this experiment several times. State a general principle which fits the results of your experiment.

10. In the figure below



$AB \overset{m}{=} DC$ and $BC \overset{m}{=} AD$. Prove that BC is parallel to AD by :
(a) identifying congruent triangles;
(b) telling why they are congruent;
(c) showing what angles have equal measures as a result of the congruence, and
(d) indicating why this enables us to conclude that the opposite sides of the quadrilateral are parallel.
Does your result agree with the principle which you discovered experimentally in exercise 9?

11. Make a statement by interchanging the if-then parts of statement (b) page 16.

12. Check whether or not the statement you have made in exercise 11 is probably true by drawing several quadrilaterals in which the if-part of that statement is true and measuring other parts.

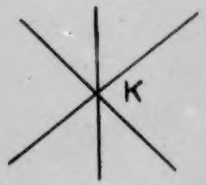
13. A square is a special kind of a parallelogram. The measures of all sides of a square are equal and all angles have a measure of 90 degrees. Show whether or not the statement is true: The diagonals of a square are perpendicular to each other. Can you use Principle 6?

14. A rectangle is a special kind of parallelogram in which the angles are right angles. Show whether or not the following statement is true in a rectangle in which adjacent sides are not equal: The diagonals of a rectangle are perpendicular to each other.

15. Draw about six rectangles of different shapes and write statements about properties of the rectangles which appear to be true. See if you can then find reasons why your statements are true or not true.

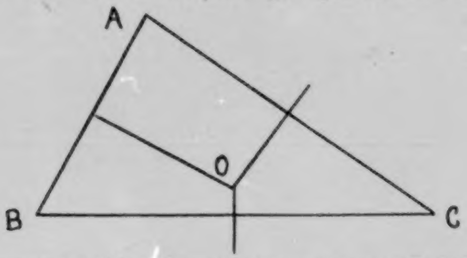
CONCURRENT LINES

Three or more lines on a point are said to be concurrent lines. The figure shows three concurrent lines on the point K.



A number of sets of concurrent lines are associated with a triangle. For example, the perpendicular bisectors of the sides of a triangle are concurrent. Let us consider why this is true.

In the figure, O is the point of intersection of the perpendicular bisector of BC and the perpendicular bisector of AC.



Since O is on the perpendicular bisector of BC, OB $\overset{m}{=}$ OC.

Since O is on the perpendicular bisector of AC, OA $\overset{m}{=}$ OC.

We conclude that OB $\overset{m}{=}$ OA. We have shown that if OB $\overset{m}{=}$ OA, then O is on the perpendicular bisector of AB. Therefore the perpendicular bisector of AB lies on point O which is the intersection of the other two perpendicular bisectors. The three perpendicular bisectors are concurrent.

EXERCISES

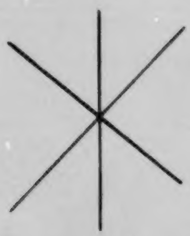
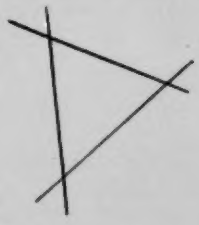
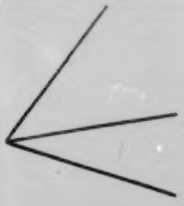
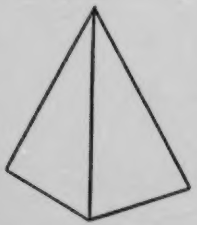
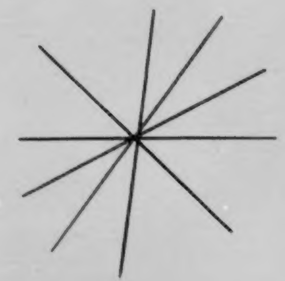
1. Draw the following:

   (a) 3 concurrent lines
   (b) 4 concurrent lines
   (c) 5 concurrent lines

2. Draw three concurrent rays such that the endpoint of the rays is the only element in the intersection set of the three rays.

3. How many angles are formed by the rays in exercise 2?

4. Which of the following appear to be concurrent lines or rays? In each figure consider all lines shown.

Use a ruler and protractor to draw a triangle with sides lengths 4 inches and 5 inches, and the included angles of measure 60°. Draw the perpendicular bisectors of the 3 sides of the triangle. Do the perpendicular bisectors appear to be concurrent.

6. Repeat exercise 5 with three triangles of different sizes and shapes. You might use a right triangle, an equilateral triangle, and a triangle with one angle greater than 90 degrees in measure. In each, construct the perpendicular bisector of each side.

7. Use ruler and compass to construct other triangles as in exercises 5 and 6 and the perpendicular bisectors of their sides.

8. In a triangle RST, the side ST is said to be opposite vertex R. If the perpendicular line from R to ST intersects ST or ST extended in point V, the segment RV is called the altitude of the triangle RST from the point R.

   Use ruler and protractor to draw a triangle of sides of length 3 inches and 4 inches, and included angle of measure 60 degrees. Draw the three altitudes of the triangle.

9. What did you discover about the altitudes in the triangle which you drew in exercise 8? Draw two other triangles and the altitudes of the triangles to see if what seemed to be true in exercise 8 holds for your other triangles.

COMPARISON OF SQUARES.

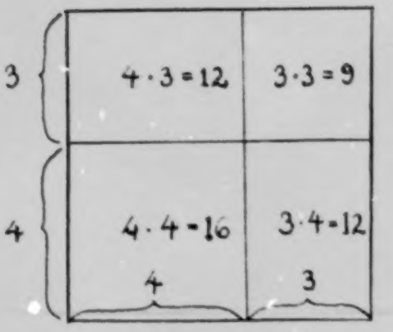


Figure 1

Figure 1 represents a square each side of which measures 7 units. The square has been divided into 4 parts which include:

(a) a square with area measure

$$(4)(4) = 16$$

(b) a square with area measure

$$(3)(3) = 9$$

(c) Two rectangles with area measure $(4)(3) = 12$.

Let us consider a second square equal in area measure but divided into parts in a different way. (If you like, you can think of only one square and division of it in two ways.) Figure 2 represents this second square. This square has been drawn so that

(i) the measures of AF, BG, CH and DE are 4 units.

(ii) the measures of FB, GC, HD, and EA are 3 units

(iii) the four triangles are congruent right triangles, since each of them has two sides which measures 4 units and 3 units, and the included angle for each is a right angle, or an angle of measure 90 degrees.

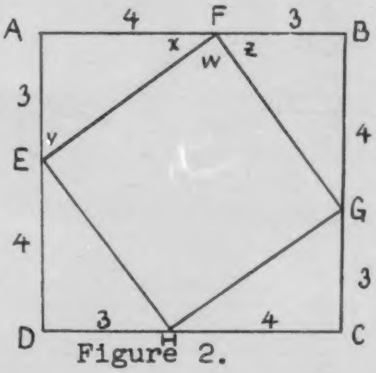


Figure 2.

EXERCISES.

1. Draw a figure like Figure 1 using inches for the units of length in your drawing.
   (a) What is the area of the square you have drawn?
   (b) What are the areas of the two smaller squares in the large square that you have drawn?
   (c) What are the areas of the rectangles you have drawn?

2. Draw a figure like Figure 2 using inches for the units of length in your figure. Label the points in your figure like those in Figure 2.

3. Give the steps in the argument to show that triangles AEF and BFG in figure 2 are congruent.

4. If the statement (iii) about figure 2 above is true, then EF $\overset{m}{=}$ FG $\overset{m}{=}$ GH $\overset{m}{=}$ HE. Why?

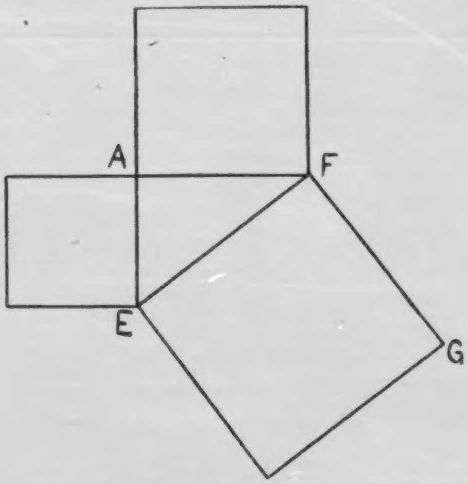
5. Copy the following and complete each statement:
In figure 2 (or exercise 2), ∠ v $\overset{m}{=}$ ∠ z since triangles
__________ and __________ are congruent. The sum of the measures of ∠ v, ∠ x, and ∠ EAF is __________ degrees. The sum of the measures of ∠ v and ∠ x is __________ degrees. The sum of the measures of ∠ z and ∠ x, and ∠ w is __________ degrees. Since ∠ v $\overset{m}{=}$ ∠ z, the sum of the measures of ∠ z and ∠ x is __________ degrees. So the measure of ∠ w is __________ degrees.

6. Using results of exercises 4 and 5, what kind of a quadrilateral is EFGH?(What are the measures of angles FGH, GHE, and HEF?)

7. Cut the square you drew in exercise 1 out of the paper. Then cut each of the two rectangles into two equal triangles. Can you fit these triangles on top of the triangles drawn in exercise 2? Why are these congruent triangles?

8. We have seen in exercise 7 that the sum of the measures of the areas of the two rectangles in figure 1 or exercise 1 are equal to the sum of the measures of the area of the four right triangles in figure 2 or exercise 2. What can you conclude about the sum of the measures of the areas of the two squares in figure 1 (exercise 1) and the measure of the area of the quadrilateral EFGH in figure 2 (exercise 2)? Remember that the two large squares are congruent and the measures of their areas is 49.

9. Place the smallest square that you cut out in exercise 7 along AE (your exercise 2) and the larger square along AF as shown in the figure. Make a statement about the squares of the lengths of the two shorter **sides** of the triangle AFE and the square of the length of the side EF, using your conclusion in exercise 8.



10. Would the same reasoning work if the sides of the triangle AEF had some other lengths? Suppose AE m 5 inches and AF m 12 inches. What is the length of EF?

In a right triangle the side opposite the right angle is called the hypotenuse. Since the reasoning in the above exercises applies to any right triangle (see exercise 10), we can state the following general principle:

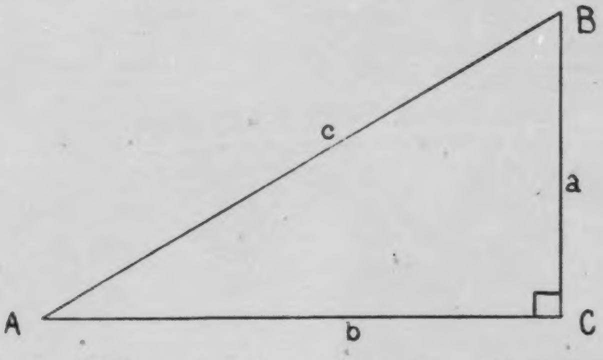
Principle 7: In a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

For example, in the above exercises the lengths of AE and AF are 3 and 4 inches respectively. If c is the length of EF, then

$$c^2 = 3^2 + 4^2 = 9 + 16 = 25$$

Therefore c must be a number which, when multiplied by itself, gives 25, so that c must equal 5. Measure EF and check this prediction.

In the right triangle below, c is the length of the hypotenuse and a and b are the lengths of the other two sides.



In each line of the following table two of the three sides of a right triangle are given, and you are to find the length of the third side.

In each case, consider c the hypotenuse.

	a	b	c
a.		12	13
b.		24	25
c.		15	17
d.		21	29
e.		35	37
f.	9		41
g.	28	45	
h.	11		61
i.		33	65
j.	16	63	
k.		55	73
l.		56	85
m.	36	77	
n.	39		89
o.		72	97
p.		99	101

You may find a table of squares useful in solving these problems. When you have filled in the empty spaces in this table, you will have a list of right triangles whose sides are natural numbers.

Such a list has been found on a Babylonian cuneiform tablet dating from between 1600 and 1900 B.C., and now in the Columbia University Library. This shows that the ancient Babylonians must have known principle 7. Since no practical applications were given, it indicates that almost 4000 years ago some people were working on mathematics for the sheer fun of it. There is no evidence that the Babylonians had any idea of what is meant by a logical proof, and it is believed that they may have found the method for constructing this list by observation and experiment.

Much later the Greek mathematician Pythagoras (when and where did he live?) discovered a proof of this principle. Plutarch (what is he famous for?) tells the story that Pythagoras was so thrilled by the discovery that "he offered a splendid sacrifice of oxen".

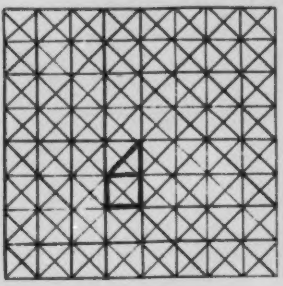
The converse of this principle is also true:

Principle 8: If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle.

EXERCISES

1. It has been said that Pythagoras noticed the right triangle property by looking at a mosaic like the following:

   Copy the mosaic design and mark with colored pencil a triangle which shows the right triangle property. As a guide a triangle and the square on one side of it have been marked in the figure.



2. As an argument to support the right triangle principle, exercise 1 takes care only of a special case. What is the special case?

3. Show for the following numbers that the square of the first is the sum of the squares of the others in each set of 3:
   (a) 5, 4, 3 (b) 13, 5, 12
   (c) 25, 7, 24 (d) 20, 16, 12

4. Draw or construct triangles with the sides of lengths given in the parts of (a) and (b) of exercise 3. Use your protractor to show that these triangles are right triangles.

5. Draw right triangles the lengths of whose shorter sides are:
   (a) 1 and 2 (b) 4 and 5 (c) 2 and 3.

Measure with a ruler to the nearest one-tenth of a centimeter, if possible, the lengths of the hypotenuses of these triangles. (If you do not have a ruler marked in centimeters, measure to the nearest 1/8 inch.)

6. Use the right triangle principle to find the squares of the lengths of the hypotenuses of the triangles in exercise 5 and check whether principle 6 is true in these cases.

7. In using unsigned numbers, we say that square root of 25 is 5 and the square root of 16 is 4. We would say that square root of 5 is a number such that its square is 5. We know that there is no fraction or whole number of which this is true. We can, however, find a number for which this is true. The approximate values of square roots of some of the whole numbers are given in a table of square roots. Use a table of square roots to find approximate values of the square roots of the following:
(a) 5 (b) 41 (c) 13.

8. Compare the results in exercises 5 and 7. Explain, using the property of right triangles why we might expect the result of exercises 5 (a) and 7 (a) to be about the same. Do the same for the parts (b) and (c) of exercises 5 and 7.

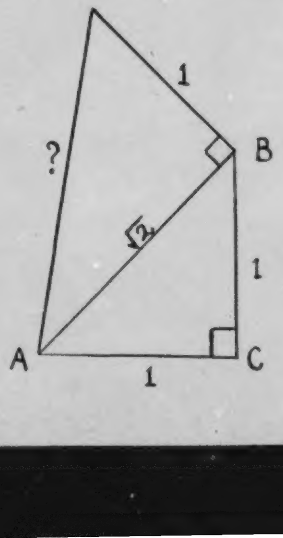
9. Use the property of right triangles to find the lengths of the hypotenuses of right triangles with sides given of the following lengths:
(a) sides of length 3 units and 5 units
The square of the length of the hypotenuse is

$$3^2 + 5^2 = 34.$$

The length of the hypotenuse is the square root of 34. From the table we find that this is 5.8, correct to the nearest one-tenth.
(b) 5 and 6 (c) 10 and 11 (d) 1 and 3.

*10. Draw a square whose sides are of length 1 unit. What is the length of the diagonal? Check by measurement. Now draw a right triangle with the sides 1 unit long. What is the length of the hypotenuse?

*11. Now draw a right triangle of sides "square root of 2" and 1 unit in length as shown in the figure.
In the figure the length of AB is the square root of 2. What is the length of the hypotenuse of this new triangle?



*12. Continue what you have started in exercise 11, and obtain line segments of the following lengths, using your unit:
(a) square root of 3 (b) square root of 4
(c) square root of 5 (d) square root of 6

Your drawing should then look like a spiral curve.

STATEMENT OF PRINCIPLES.

Principle 1: Corresponding segments and corresponding angles of congruent figures have equal measures.

Principle 2: If two angles and the included side of one triangle are equal in measure, respectively, to two angles and the included side of another triangle, then the triangles are congruent.

Principle 3: If the three sides of one triangle are equal in measure, respectively, to the three sides of another triangle, then the triangles are congruent.

Principle 4: If two sides and the included angle of one triangle are equal in measure, respectively, to two sides and the included angle of another triangle, then the triangles are congruent.

Principle 5: If the point P is on the perpendicular bisector of the line segment AB, then the distance from P to the end points A and B are equal.

Principle 6: If the distances from the point P to the points A and B are equal, then P is on the perpendicular bisector of the line segment AB.

Principle 7: In a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

Principle 8: If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other sides, then the triangle is a right triangle.

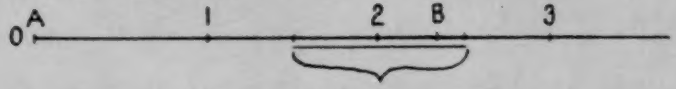
UNIT X

Measurement and Approximation

When you use numbers to count separate objects you need only whole numbers, which are the same as what mathematicians call the natural numbers or the positive integers. When you count the number of people in a room you know the result will be an integer; there will be exactly 11, not 11 1/4 or 10 1/2. If there are a great many people, or you are not sure you have counted correctly, you may say there are "about 330," rounding the number to the nearest hundred.

When you measure something the situation is different. When you have measured the length of a line segment with a ruler divided into quarter-inches, the end of the segment probably fell between two quarter-inch marks, and you had to judge to which mark it was closer. Even though the end seemed to fall almost exactly on a quarter-inch mark, if you had looked at it through a magnifying glass you would probably have found that there was a difference. And if you had then changed to a ruler with the inches divided into sixteenths, you might have decided that the end of the segment was nearer to one of the sixteenth-inch marks than to the quarter-inch mark. Furthermore, as you know, between two points on a line there is always a third point. So scientists and mathematicians agree that measurement of this sort cannot be considered exact, but only approximate; and that the important thing is to know just how inexact a measurement is, and to state it so that other people will also know from the way it is stated how exact it is.

Look at the line below, which shows a scale divided into one-inch units. The 0 point is labeled "A", and point B is between the 2-inch mark and the 3-inch mark. Since B is clearly closer to the two-inch mark, we say that segment AB is 2 inches. The fact that we state the measurement "2 inches" implies that AB was measured to the nearest inch. However, any point which is more than 1 1/2 inches from A and less than 2 1/2 inches from A would be the endpoint of a segment which, to the nearest inch, is also 2 inches. The mark below the line shows the space within which the endpoint of a line segment 2 inches long (to the nearest inch) might fall. Such a segment might actually be almost 1/2 inch less than 2, or almost 1/2 inch more than 2 inches. We therefore say that, when a line segment is measured to the nearest whole inch the "greatest possible error" is 1/2 inch. This does not mean that you have made a mistake (or that you have not). It simply means that, if you measure correctly to the nearest whole inch, any measurement more than 1 1/2 inches and less than 2 1/2 inches will be reported in the same way, as 2 inches. Consequently such measurements are sometimes stated as 2 ± 1/2. (The symbol "±" is read "plus or minus.")



Exercises

1. Draw a line and mark on it a scale with divisions of 1/4 inch. Mark the zero point C, place a point between 1 2/4 and 1 3/4, but closer to 1 3/4, and call the point D. How long is CD, but to the nearest 1/4 inch?
2. Between what two points on the scale must D lie if the measurement, to the nearest 1/4 inch, is to be 1 3/4? How far from 1 3/4 is each of these points?
3. When you measure to the nearest 1/4 inch, what is the "greatest possible error?"
4. Why may this measurement of CD be stated "1 3/4 ± 1/8?"
5. a. The measurement of a line segment was stated to be 1 1/8 inches. This segment must have been measured to the nearest _______ of an inch.

   b. The endpoint of the segment must have fallen between 1_____ and 1 _____.

   c. The "greatest possible error" in the measurement of this segment is _____.

   d. The measurement might be stated as 1 1/8 ± _____.
6. The measurement of a line segment was stated as 2 5/16 ± 1/32. What is the range within which the end of this segment must lie?

7. When a line segment is measured to the nearest inch, we say the unit used is 1 inch. When it is measured to the nearest 1/2 inch, we say the unit is 1/2 inch. If a measurement is stated to be 5/16 inch, this means the measurement was made to the nearest _____ of an inch, so the unit is _____ inch.
8. The "greatest possible error" in a measurement is always what fractional part of the unit used?
9. Often an inch scale is divided into tenths of an inch. A line segment was measured with such a scale, and stated to be 3 7/10 inches. What was the unit of measurement? What was the greatest possible error? State the measurement 3 7/10 ± _____.
10. Machinists sometimes measure to the nearest 1/100 of an inch. What is the greatest possible error in such a measurement?

Precision

Consider the two measurements, 10 1/8 inches and 12 1/2 inches. What unit was used for each measurement? What is the greatest possible error of each measurement? Since the unit for the first measurement is 1/8 inch, and the unit for the second measurement is 1/2 inch, we say that the first measurement is more precise than the second, or has greater precision. Notice also that the greatest possible error of the first measurement is 1/2 of 1/8 or 1/16 inch, and for the second measurement if 1/2 of 1/2 inch, or 1/4 inch. The first measurement has a smaller possible error than the second. So the more precise of two measurements is the one made with the smaller unit, and for which the greatest possible error is therefore the smaller.

It is very important that measurements be stated so as to show correctly how precise they are. If you have measured a line segment to the nearest 1/8 inch, and the measurement is 2 6/8 inches you should not change the fraction to 3/4, for that would make it appear that the unit was 1/4 inch, rather than 1/8 inch. If you measured to the nearest 1/4 inch, and the measurement was closer to 3 inches than to 2 3/4 or 3 1/4 inches you should state it to be 3 0/4, so that other people will know that the unit used was 1/4 inch.

Exercises

1. Suppose you measured a line to the nearest hundredth of an inch. Which of these numbers would state the measurement best?

   3.2 inches   3.20 inches   3.200 inches

2. Suppose you measured to the nearest tenth of an inch. Which of these numbers should you use to state the result?

   4 inches   4.0 inches   4.00 inches

3. For each of the measurements below tell the unit of measurement, and the greatest possible error. Then tell which measurement in each pair has the greater precision.

   a. 5.2 feet, 2 1/4 feet
   b. .68 feet, 23.5 feet
   c. .235 inches, .146 inches

4. What is your age to the nearest year, that is, what is your nearest birthday - tenth, eleventh, twelfth,...? All of you who say "13" must be between _____ and _____ years old.

5. What is your age to the nearest 1/2 year? All of you who say "12 1/2" must have birthdays between the months of _____ and _____.

Usually scientific measurements are expressed in decimal form. For instance, it is known that one meter (a unit in the metric system of measures) is about 39.37 inches. This means that a meter is closer to 39.37 inches than it is to 39.38 inches or 39.36 inches. In other words, one meter lies between 39.375 inches and 39.365 inches.

In the case above one can tell from the way the number is written how precise it is supposed to be. But if we were told that something is 37800 feet long, it is not clear whether the zeroes at the end are just to keep the decimal place where it belongs or to indicate the precision. The unit of measurement may have been 1 foot, 10 feet, or 100 feet. Various interpretations might be: (1) the length is closer to 37800 than to 37900 or 37700; (2) the length is closer to 37800 than 37810 or 37990; or (3) the length is closer to 37800 than 37801 or 37999. In a case like this, we frequently underline a zero to show how precise the measurement is. For example, 378$\underline{0}$0 means that the measurement is precise to the nearest 10 feet, while 3780$\underline{0}$ means it was made to the nearest foot. If neither zero is underlined, we understand that the measurement was made to the nearest 100 feet. If a measurement is stated as 5.640 feet we understand, without underlining the zero, that the unit is one-thousandth of a foot, for otherwise the zero would not be written at all.

6. For each measurement below tell what unit of measurement was used and the greatest possible error.

   a. 52700 ft. b. 527$\underline{0}$ ft. c. 527$\underline{0}$0 ft.

   d. 52.7 ft. e. .5270 ft. f. 527.0 ft.

7. Which of the measurements in Ex. 6 is most precise? Which is least precise? Do any two measurements have the same precision?

8. Show by underlining a zero the precision of the following measurements:

   a. 4200 ft., measured to the nearest foot.

b. 23000 miles, measured to the nearest hundred miles.

c. 48,000,000 people, reported to the nearest ten-thousand.

Relative Error

While two measurements may be made with the same precision (that is, with the same unit) and their greatest possible error is therefore the same, this error is more important in some cases than in others. An error of 1/2 inch in measuring your height would not be very misleading, but an error of 1/2 inch in measuring your nose would be misleading. We can get a measure of the importance of the possible error by comparing it with the measurement. Consider these measurements and their possible errors:

$$4 \text{ in.} \pm .5 \text{ in.}$$

$$58 \text{ in.} \pm .5 \text{ in.}$$

Since these measurements are both made to the nearest inch, the greatest possible error in each case is .5 inch. But .5 is a much larger fraction of 4 than it is of 58. If we divide the possible error by each of the measurements we get these results:

$$\frac{.5}{4} = \frac{5}{40} = .125, \text{ or } 12.5\%$$

$$\frac{.5}{58} = \frac{5}{580} = .0086, \text{ or } .86\%$$

We see that in the first case the possible error is 12.5% of the measurement, and in the second the possible error is less than 1% of the measurement.

The per cent of the greatest possible error is of the measurement is called the *per cent of error* or *relative error* of the measurement.

Exercises

1. State the greatest possible error for each of these measurements.
   a. 52 ft. b. 4.1 in. c. 2580 mi. d. 36$\underline{0}$ ft.
   e. 7.03 in. f. .006 ft. g. 54,000 mi. h. 54,0$\underline{0}$0 mi.
2. Find the per cent of error, or relative error, of each measurement in Exercise 1.
3. Find the greatest possible error and the relative error for each of the following measurements.
   a. 9.3 ft. b. .093 ft. c. 930 ft. d. 93,000 ft.
4. What do you observe about your answers for Exercise 3? Can you explain why the relative errors, or per cents of error, should be the same for all of these measurements?

Significant Digits (Accuracy)

Look again at Exercise 3 above. Notice that in Part a, 9.3 feet, the unit of measurement is .1 foot, so the measurement might be written as follows: 9.3 ft. = 93 x .1 ft.

In 3 b, .093 ft. = 93 x .001 ft.
In 3 c, 930 ft. = 93 x 10 ft.
In 3 d, 93,000 ft. = 93 x 1000 ft.

The units are different, but in each case the number of units is 93. We call the 9 and 3 in 93 *significant digits*. In measurements, the figures in the numerals which indicate the number of units in the measurement are *significant digits*.

Exercises

1. Write each measurement below to show the unit of measure and the number of units. Then write the significant digits.

   a. 520 ft.   b. 32.46 in.   c. .002 in.   d. 403.6 ft.

   e. $25,8\underline{0}0$ ft.   f. .0015 in.   g. 38.90 ft.   h. .0603 in.

2. Find the relative error for the following measurements.

   a. 26.3 ft.   b. .263 ft.   c. 2630 ft.

   d. 51,000 mi.   e. 5.1 ft.   f. .051 in.

3. What is the precision of each measurement in Exercise 1 and 2?

The per cent of error of a measurement, or its relative error, shows its *accuracy*. The smaller the per cent of error, the greater is the *accuracy* of the measurement.

4. Which of the measurements in Exercise 2 have the same accuracy? Which have the same significant digits?

5. How many significant digits are there in each of these measurements?

   a. 52.1 in.   b. 52.10 in.   c. 3.68 in.   d. 368.0 in.

6. Find the relative error of each of the measurements in Exercise 5.

7. From your answers for Exercise 5 and 6, can you see any relation between the number of significant digits in a measurement and its relative error?

8. Without computing, can you tell which of the measurements below has the greatest accuracy? Which is the least accurate?

23.6 in. .043 in. 7812 in. .2 in.

Adding and Subtracting Measurements

Since measurements are never exact, the answers to any questions which depend on those measurements are also approximate. For instance, suppose you measured the length of a room by making two marks along a wall, call them A and B, then measured the distance from the corner to A, from A to B, and then from B to the other corner. Measurements which are to be added should all be made with the same precision. Suppose, to the nearest fourth of an inch, the measurements were 72 1/4 inches, 40 2/4 inches, 22 3/4 inches. You would add these amounts to get 135 2/4 inches. Of course, the distances might have been shorter in each case. They could have been almost as small as 72 1/8, 40 3/8, and 22 5/8 in which case the distance would have been almost as small as 135 1/8 inches, which is three-eighth of an inch less than 135 2/4. Also, each distance might have been longer by nearly one-eighth of an inch, in which case the total length might have been almost three-eights of an inch longer than 135 2/4. The error of a sum could be the sum of the possible errors, but usually the errors will to a certain extent cancel each other, one measurement being too long and another too short.

In general, therefore, the sum of several measurements, all made with the same precision, has the same precision as the measurements which were added. That is, if all the measurements added are precise to the nearest tenth of an inch, their sum (or difference) is also precise to the nearest tenth of an inch.

Sometimes measurements to be added or subtracted have not been made with the same precision. One may have been made to the nearest inch, another to the nearest fourth-inch, and so on. In that case, their sum (or difference) is only as precise as the least precise of the measurements. For example, 5 3/8 in. + 2 1/4 in. + 3 1/2 in. gives a sum of 11 1/8 in. But since the least precise measurement was made with a unit of 1/2 inch, the sum is precise only to the nearest half-inch. The sum should therefore be rounded to 11 0/2 inches.

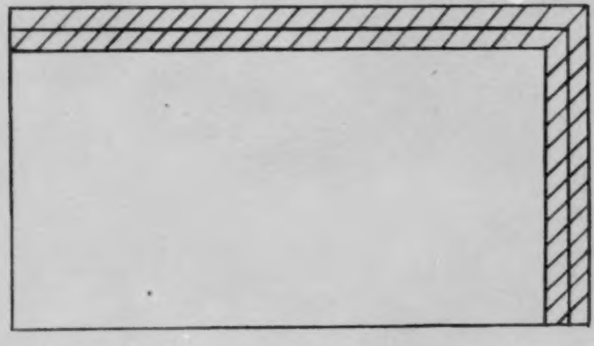
Exercises

With what precision should the sums of these measures be given?

1. 5 1/2 in. + 6 1/2 in. + 3 0/2 in.
2. 3 1/4 in. + 6 1/2 in. + 3 in.
3. 4.2 in. + 5.03 in.
4. 42.5 in. + 36.0 in. + 49.8 in.
5. .004 in. + 2.1 in. + 6.135 in.
6. 2 3/4 in. + 1 5/16 in. + 3 3/8 in.

Approximate Measurement of Area

You know that the area of a rectangle is found by multiplying the number of units in the length by the number of the same units in the width. Suppose that the dimensions of a rectangle are 3 1/4 inches and 1 3/4 inches. Since the measuring was done to the nearest 1/4 of an inch, the measurements can be stated as 3 1/4 ± 1/8 and 1 3/4 ± 1/8. This means that the length might be almost as small as 3 1/8 inches and the width almost as small as 1 5/8 inches. Or the length might be almost 3 3/8 inches and the width almost 1 7/8 inches. Look at the drawing to see what this means. The outside lines show how the rectangle would look if the dimensions were as large as possible. The inner lines show how it would look if the length and width were as small as possible. And the shaded part shows the difference between the largest possible area and the smallest possible area with the given measurements.



To figure out these areas, we multiply 3 1/8 x 1 5/8 to find the smallest possible area, and 3 3/8 x 1 7/8 to find the area if it is as large as possible.

$$3\ 1/8 \times 1\ 5/8 = \frac{25}{8} \times \frac{13}{8} = \frac{325}{64} = 5\ 5/64$$

$$3\ 3/8 \times 1\ 7/8 = \frac{27}{8} \times \frac{15}{8} = \frac{405}{64} = 6\ 21/64$$

These results show that there seems to be a difference of more than 1 square inch in the two possible areas.

If we find the area by using the measured length and width, we find that

$$3\ 1/4 \times 1\ 3/4 = \frac{13}{4} \times \frac{7}{4} = \frac{91}{16} = 5\ \frac{11}{16}$$

However, since we have seen that the area might be either larger or smaller than this number of square inches, it would not be correct to give the result in this way, which means that the area has been found to the nearest 16th of a square inch. The area could be as much as 41/64 square inches greater or 39/64 square inches less than 5 11/16. Since both 41/64 and 39/64 are about 40/64, which is 5/8, we could express the area as 5 11/16 ± 5/8 square inches. One good way of expressing the result to indicate the accuracy would be to write 5 1/2 square inches since the area cannot be less than 5 and not much more than 6.

Exercise

1. Suppose a rectangle is 2 1/2 inches long and 1 1/2 inches wide. Make a drawing of the rectangle. Show on the drawing that the length is 2 1/2 ± 1/4 and the width 1 1/2 ± 1/4. Then find the largest area possible and the smallest area possible, and find the difference, or uncertain part. Then find the area with the measured dimensions, and find the result to the nearest 1/2 square inch.

We can see a little better what is happening in general if we do the problem above somewhat differently. We noticed that the dimensions of the rectangle might be as big as (3 1/4 + 1/8) and (1 3/4 + 1/8). Suppose we multiply these numbers without combining. (We are using the distributive property). Then we would have

$$\begin{aligned}(3\ 1/4+1/8)(1\ 3/4+1/8) &= (3\ 1/4+1/8)(1\ 3/4)+(3\ 1/4+1/8)(1/8)\\ &= (3\ 1/4\times1\ 3/4)+(1/8\times1\ 3/4)+(3\ 1/4\times1/8)+(1/8\times1/8)\\ &= (3\ 1/4\times1\ 3/4)+(1/8\times1\ 3/4)+(1/8\times3\ 1/4)+(1/8\times1/8)\\ &= (3\ 1/4\times1\ 3/4)+(1/8)(1\ 3/4+3\ 1/4)+(1/8\times1/8)\end{aligned}$$

The first product is just the product of the approximate measurements. The second product is the amount of possible error multiplied by the sum of the approximate measurements. The last product is very small. Hence the error in the number of square units of the computed area is just about the product of the possible error in the linear units multiplied by the sum of the approximate lengths. To check this against our previous computation notice that $(1/8)(1\ 3/4 + 3\ 1/4)$ $= (1/8)(20/4) = 5/8$.

We noticed in the example which we just finished that the error in the number of square units of the computed area is just about the same as the product of the possible error in the linear units (assuming the linear measurements are of the same precision) multiplied by the sum of the approximate lengths. One can see that this would be true for any numbers by seeing how the above example works. A slightly better way of seeing this is to use letters in place of the numbers. We could use the letter a in place of 3 1/4, and the letter b in place of 1 3/4. Then the measured length of the rectangle would be a inches and the width would be b inches. The computed area would be the product of a and b which is usually written ab. Then if the possible error is 1/8, the dimensions could be as large as $(a + 1/8)$ and $(b + 1/8)$. Then the area could be as large as

$(a + 1/8)(b + 1/8) = ab + (1/8)b + a(1/8) + (1/8)(1/8)$.

Since a and b are numbers, $a(1/8)$ is equal to $(1/8)a$. Then, using the distributive property, we have

$(a + 1/8)(b + 1/8) = ab + (1/8)(a + b) + (1/8)(1/8)$.

The product ab is just the product of the approximate measurements. The next product $(1/8)(a + b)$ is the amount of possible error in the linear measurements multiplied by the sum of the approximate measurements. The product $(1/8)(1/8)$ is very small. Thus we have the same general conclusion that we had before. Furthermore, we could replace a and b by any other measurements and the general result would be the same.

Exercises

1. If instead of considering how large the dimensions of the rectangle in the above example could be, we considered how small they could be, that is $(3\ 1/4 - 1/8)$ and $(1\ 3/4 - 1/8)$, find the product as we did above and draw similar conclusions.
2. Suppose the lengths of the sides of a rectangle are measured to be 46.2 and 23.4 inches precise to the nearest tenth; that is, that the greatest possible error is .05. About how large is the possible error in the area found by multiplying the measurements of the sides?

3. Suppose the lengths of the sides of a rectangle are measured to the nearest tenth of an inch and that each length is less than 100 inches. By how much may the true value of the area differ from the computed value?
4. If c and d represent two measurements given with a greatest possible error of .1, what is the possible error of the product of c and d?
5. Answer the previous question if the greatest possible error is .01 instead of .1.

Significant Digits in a Product

You have seen that when an area is computed by multiplying two linear measurements which have the same precision, you can find out about what the greatest possible area is. Scientists make use of significant digits to decide how such products should be stated.

Suppose a rectangle has the dimensions 3.4 inches and 2.86 inches. You know that these measurements are really 3.4 in. ± .05 in. and 2.86 in. ± .005 in. So the 4 in 3.4 and the 6 in 2.86 are both approximate, not exact. The multiplication of the two dimensions is shown below in two ways.

$$\begin{array}{r} 2.86 \\ 3.4 \\ \hline 1\;1\;4\;4 \\ 8\;5\;8\; \\ \hline 9.7\;2\;4 \end{array} \qquad\qquad \begin{array}{r} 2.8\dot{6} \\ 3.\dot{4} \\ \hline 1\;\dot{1}\;\dot{4}\;\dot{4} \\ 8\;5\;\dot{8}\; \\ \hline 9.\dot{7}\;\dot{2}\;\dot{4}, \end{array} \qquad \text{or } 9.7$$

In the work on the left, the product 9.724 suggests that the precision of the area is to the nearest one-thousandth of a square inch. But since the original measurements on which the area is based were precise only to the nearest tenth and hundredth of an inch, such precision in stating the area is not justified.

In the work on the right, a dot (˙) is placed above the 6 in 2.86 and above the 4 in 3.4 to remind us that these figures are not exact. Then dots are placed in the products above the figures obtained from the 6 and the 4, since these figures will also be approximate. We can see, then, that in the final product the figures 7, 2, and 4 are not exact either. The product is therefore stated as 9.7. The 7 is approximate, as is the last figure in any measurement.

Notice that 9.7 has the same number of significant digits as the linear measurement (3.4) with the smaller number of significant digits. In general, scientists state the result obtained by multiplying two measurements using as many significant digits as there are in the original measurement with the smaller number of significant digits.

Exercises

1. In the multiplication below, dots are placed over the digits which are approximate in the factors. Copy the

work and put dots over the figures in the products which will also be approximate.

a.
$$\begin{array}{r} .06\dot{8} \\ \underline{4.\dot{7}} \\ 476 \\ \underline{272} \\ .3196 \end{array}$$

b.
$$\begin{array}{r} 48\dot{2} \\ \underline{9.\dot{3}} \\ 1446 \\ \underline{4338} \\ 4482.6 \end{array}$$

c.
$$\begin{array}{r} 5.1\dot{4} \\ \underline{.\dot{6}} \\ 3.084 \end{array}$$

2. Round each product in Exercise 1 to the same number of significant digits as there are in the factor with the smaller number of significant digits.

3. The numerals in the multiplications below represent number of inches.

   a. 4.96 x 3.1
   b. $28\underline{0}$ x 6.035
   c. 4600 x 3.8
   d. .5423 x .790
   e. How many significant digits are there in each factor?
   f. How many significant digits should each product have?
   g. Multiply the numbers, and write each product with the correct number of significant digits.

XI. THE SCIENTIFIC SEESAW OR MATHEMATICS AT WORK IN SCIENCE

Have you ever played on a seesaw?

If your weight is 100 pounds and your partner on the other side of the seesaw weighs 85 pounds, where does he have to sit so that the two sides will just balance? Will he be closer to the center or farther away from it than you? How far?

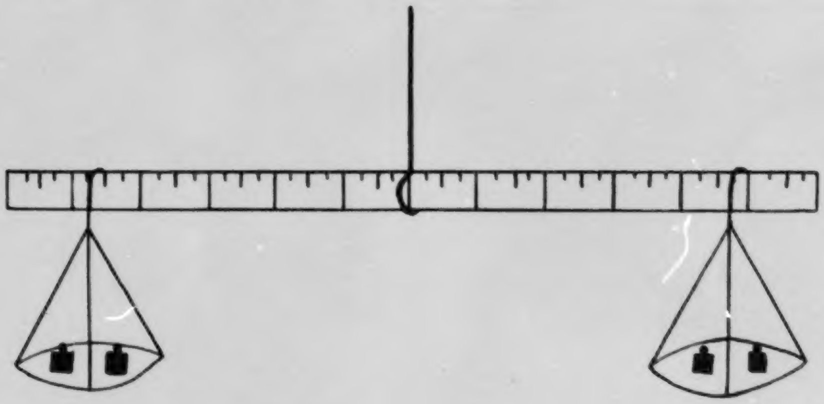
The seesaw is a form of simple machine that is used a great deal by scientists. It belongs to the family of machines called "levers." Scientists have investigated how it works, how to balance weights on it, and they have expressed their findings in a mathematical formula. Levers are used in very many ways.

Today you will be the scientist. You will set up the equipment, make the observations, discover the rule or law, and state it in mathematical form.

The following experiment is one example of how a scientist makes certain observations in the laboratory, studies them mathematically and draws conclusions from them, states the conclusions by means of a mathematical formula, makes predictions, and then goes back to the laboratory to test whether the formula works in any similar situation.

EXPERIMENT

To begin to study this scientific seesaw, your equipment will look something like this:



Materials:

A meter stick or yardstick
String and 2 pans or bags to hold the weights
A set of metric weights. (If a set of weights is not available, a batch of pennies can be used.)

Procedure

1. Suspend the bar by tying a string somewhere near the middle.

2. Place one weight on one side of the fulcrum and another weight on the other side, and try to make the lever balance. Try some other weights and make them balance. Do you find that you have to change the distance according to the size of the weights?

Note: Scientists do not usually make their discoveries with haphazard trials as you have just been doing, but only after they have set up a very carefully planned experiment.

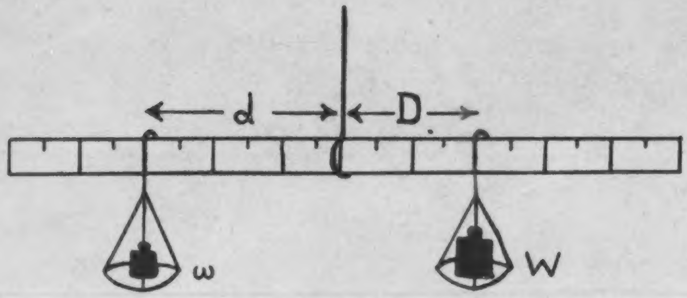
Let us set up a plan and see what we can discover about the lever with the aid of mathematics.

3.(a). Hang a weight of 10 grams (or 10 pennies if you do not have metric weights) at a distance of 12 centimeters from the fulcrum, and balance it with a 10 gram weight on the other side. Observe the distance of this second weight when the lever is in balance and record the distance in a table, column (a), similar to the one below.

TABLE I

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
w =	10	10	10	10	10	10	10
d =	12	12	12	12	12	12	12
W =	10	20	5	8	15	24	12
D =							

Note: w and d represent the weight and distance respectively on one side of the fulcrum, W and D the weight and distance on the other side of the fulcrum.



(b) Now double the weight of W (make it 20 grams) and find where it must be placed to balance weight w. Write the distance in your table under "20," column (b).

(c) Make weight W only half as large as it was in the first case (make it 5 grams), adjust the balance and read the distance from the fulcrum. Write it in the table, column (c) under "5."

(d) Notice that in these first three trials, weight w and its distance from the fulcrum remained the same, and changes were made in weight W. Make at least three or four other changes and write the results in your table as in Table I, columns (d), (e), (f), and (g).

4. Now, as indicated in Table II, let weight w be 16 grams and its distance from the fulcrum 6 cm. Find how heavy weight W will have to be to balance at 6 cm. on the other side of the fulcrum. What weight will balance the lever 4 cm. from the fulcrum? 16 cm? Try several other distances from the fulcrum, find what weight will just balance the lever, and fill in your Table II.

TABLE II

w	16	16	16	16	16	16	16
d	6	6	6	6	6	6	6
W							
D	6	4	16				

5. Try other weights and distances as suggested in Tables III and IV and fill in similar tables of your own.

TABLE III

w	20	40	10				
d							
W	20	20	20	20	20	20	20
D	15	15	15	15	15	15	15

TABLE IV

w	18	18	18	18	18	18	18
d	5	5	5	5	5	5	5
W	15	10	30	45			
D					10	15	18

Let us now study the tables to see if there seems to be any law which expresses all relationships and which could be expressed mathematically. If there is such a law, we could use it to predict where to place any weight to balance any other weight.

Exercises

1. (a) From Table I what do you notice between the placement of equal weights on two sides of a fulcrum?

   (b) If only weight W is doubled, how does its distance from the fulcrum change?

   (c) If weight W is made half as much, how does its distance from the fulcrum change?

   (d) State a rule that seems to hold concerning w, d, W, and D?

2. Use the laws that you found in Exercise 1 to predict the correct measures in the missing parts of this table.

TABLE V

w	25	35	12	45	21	15	23	11	?	14	10	100	100	100
d	3	4	5	3	5	?	4	8	7	5.5	50	50	500	5000
W	15	?	7	?	14	20	12	?	13	10	?	?	?	?
D	?	7	?	9	?	4.5	?	10	8	?	5	5	5	5

3. Go back to your equipment and check your results in several of the above exercises to see if there is a balance.

A Graph of the Experiment

In order to study a set of observations from an experiment, scientists often make a graph or drawing of them on a set of axes.

If you have not drawn such graphs, the following will give you the general idea.

Exercises

1. (a) Use graph paper and begin with two perpendicular lines called axes. The intersection of the axes is named point O. (See attached graph.)

   (b) One axis is named W and the other D, and we will locate the points corresponding to pairs of weights and distances in your Table I.

   In Table I, the first weight was 10 and the distance 12. Locate 10 on the W axis. Follow the vertical line through 10 to the point where it meets the horizontal line through 12 on the D axis. This point is called (10, 12).

   Similarly, from Table I, when W = 20 and D = 6, locate 20 on the W axis. Follow the vertical line through 20 to the point where it meets the horizontal line through 6 on the D axis. This point is called (20, 6).

   Locate the other points from your Table I. These are (10, 12), (20, 6), (5, 24), (8, 15) etc.

   Draw a smooth, freehand curve through the points you have located. This curve gives you the general picture of the relation between the weights and the distances.

If any point seems to lie rather far from a smooth curve through the other points, check your computation for that one. Maybe you've made a mistake in arithmetic.

2. By examining the graph, find answers to these questions:

   (a) What happens to the distance as the weight increases?

   (b) What happens to the weight as the distance increases?

3. Now that you have a graph of the relation between weight and distance, you can do what a scientist does - you can predict the value of the distance for any weight which will keep the balance.

   In order to find the value of W when D is 24, locate 24 on the D axis and follow the horizontal line through 24 until it meets the curve. Read the value on the W scale directly beneath this point. You should have 5. (The dotted line and arrows on the graph may help you.)

   Find the values of W from the graph.

   (a) D = 30 (b) D = 15 (c) D = 9

   (W = ?) (W = ?) (W = ?)

4. Tell whether or not each of the following points is on the graph.

   (a) W = 10 (b) W = 5 (c) W = 20

   D = 25 D = 5 D = 15

5. Fill in the following table by selecting a value of W on the graph and following the perpendicular through that point to the curve, thus locating the corresponding value of D on the curve.

if W	6	7	8	9	16	17	21	23	25
then D	20	?	?						

Note: Mathematicians call the curve that you have drawn a hyperbola. It is a very interesting and useful curve.

Other examples of levers

Several forms of the lever are illustrated by the following:

Pictures or Sketches of:
- (seesaw
- (crowbar
- (nutcracker
- (tongs
- (Archimedes lifting the earth with a long lever.

Archimedes once said that if he could have a long enough lever and a place to rest it, he could lift the earth. Explain.

Special projects

Many scientific facts were undiscovered for thousands of years until an alert scientist carefully set up an experiment in a manner similar to the one you have done and made discoveries on the basis of observations.

(a) For thousands of years, people assumed that if a heavy object and a light object were dropped at the same time, the heavier one would fall much faster than a light one.

Look up the story of Galileo and his experiment with falling objects and see what he discovered.

(b) From time immemorial, people watched eclipses of the sun and moon and saw the round shadow of the earth but did not discover that the earth was round. Eratosthenes, in 230 B.C., computed the distance around the world by his observations of the sun in two locations in Egypt, yet seventeen hundred years later when Columbus started on his journey, many people still believed the world was flat.

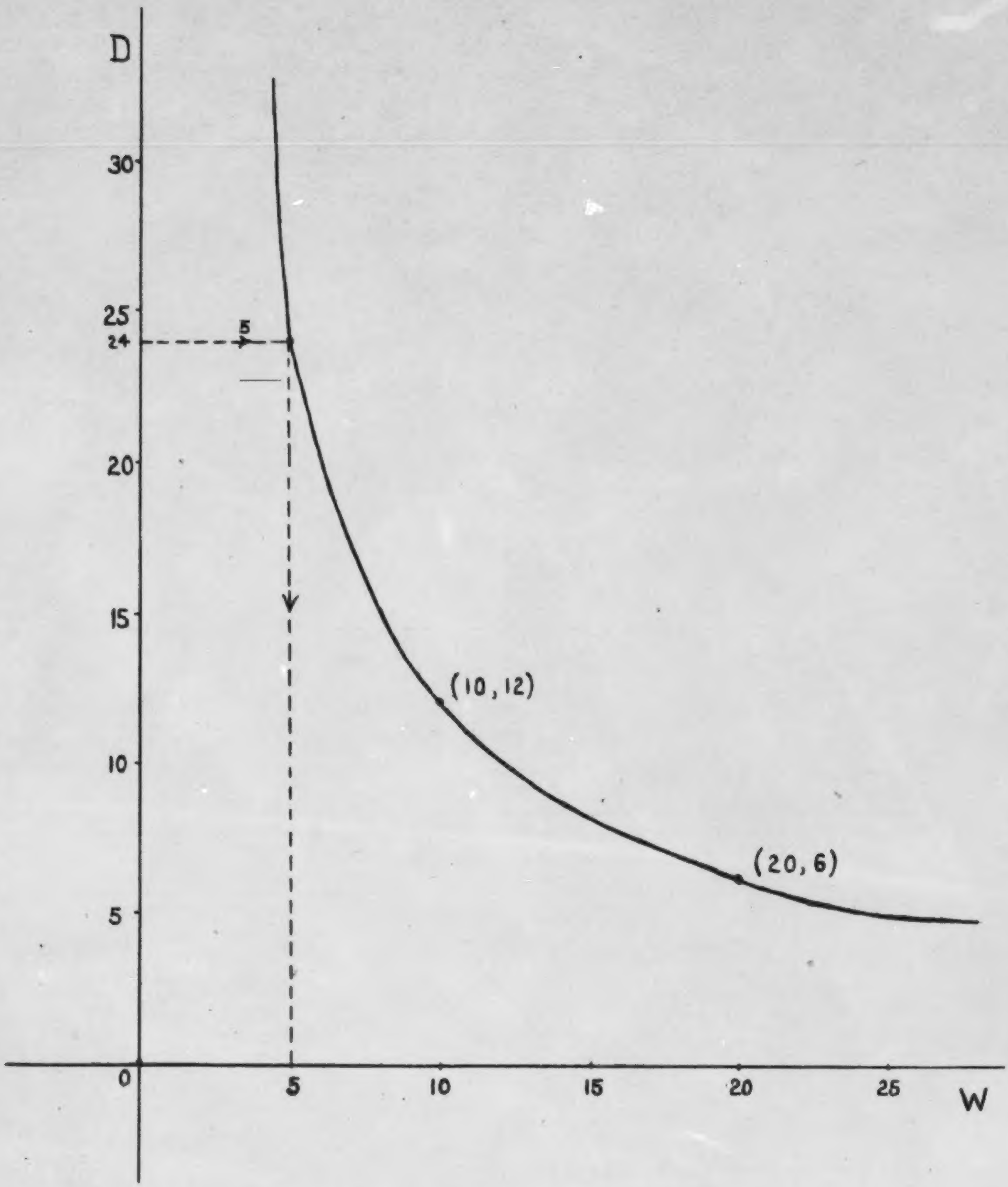
Look up in a history of mathematics book or in an encyclopedia the story of Eratosthenes and this experiment.

(c) People had watched pendulums swing for many centuries before Galileo did some measuring and calculating and discovered the law which gives the relation between the length of the pendulum and the time of its swing.

Look up this experiment in a history of mathematics book.

Notice that all such experiments are based on many careful measurements and observations in order to discover the scientific law, and then the law is stated in mathematical terms. Thus we see the dependence of science on mathematics.

The Seesaw Graph



UNIT 12

Uncle Sam as a Statistician

When you studied the Constitution of the United States, did you notice that many of the sections need a good deal of mathematics to carry them out? For instance, notice these sections in Article I:

> Sec. 2. Apportionment. Representatives and direct taxes shall be apportioned...according to their respective numbers. The actual enumeration shall be made...within every...term of ten years.
>
> Sec. 8. The Congress shall have power:
>
> To lay and collect taxes, duties, imposts, and excises, to pay the debts and provide for the common defense and general welfare of the United States.
>
> To borrow money on the credit of the United States.

When you examine the above provisions and speculate about the mathematics needed to carry them out, you will probably raise questions such as these:

(a) How does Uncle Sam go about counting 170 million people?

How does he sort all that information?

What use would he make of computing machines?

What does the Census Bureau do between censuses?

After the citizens of the United States are counted every ten years, and the information is used to decide on the number of representatives in Congress from each district, what other uses can be made of the information that has been collected?

(b) How does Uncle Sam estimate the government expense for any year and how does he know how much to collect in taxes?

How does he decide whether to lower taxes or not? Some people think that if taxes are lowered, the government will not collect enough money to pay this year's expenses; others think that if taxes are lowered, people will buy more goods, more people will make money, and the government will make more on income taxes. Who is right? Such questions cannot be decided by personal opinions or by experiment since a trial of one or the other plan might cause a great loss to the government. So the ones who decide must study the statistics and base their decisions on them.

How does Uncle Sam decide what duties to charge on foreign imports? If duties are low, will people buy foreign products rather than American-made ones? If duties are high, how will that affect American industries like a clothing industry which manufacturers its products out of imported material, or a watch assembly industry which uses foreign-made works for watches?

(c) And how does Uncle Sam decide what interest to pay on the money he borrows -- that is, on U. S. Government Bonds? If he pays a high rate of interest on his bonds, which of course are a very safe investment, will people invest their money in banks, real estate, mining companies, etc., if they pay the same rate of interest? Will such companies have to pay a higher rate of interest than Uncle Sam in order to get people to invest in them?

All such questions are answered only after a very careful study of the data or statistics that have been collected. Naturally we will not be able to answer the above questions in this unit, but we will be able to study a few ideas about statistics. In fact, the word "statistics" is derived from the word "state", for governments have always had to keep very careful records and collect many data in order to be able to make wise decisions about questions such as those that were suggested. The method of drawing conclusions or making decisions on the basis of samples of data is called "statistical inference."

The study, sorting, and handling of statistics require mathematical skills, and the fundamentals can be studied in junior high schools.

Can you list some reasons why Uncle Sam needs to have such information as:

(a) The number of men who will reach the age of 18 next year?

(b) The total national income?

(c) The number of unemployed people?

(d) The size of the labor force?

(e) The number of people who will be 65 next year and collect Social Security payments?

(f) The amount of gasoline used in a year?

List other information which you can think of which might be needed by the federal government.

The federal government has always needed to collect statistics of many kinds. Even before the first census was taken in 1790, Alexander Hamilton started collecting statistics for the Treasury Department on foreign trade. He saw that such figures would be important for the developing of new industries in this country and for expansion of foreign trade. In the third census (1810) Thomas Jefferson ordered "an account of the several manufacturing establishments and manufacturers."

A list of the agencies which collect statistics for Uncle Sam is very extensive:

The Organization of the Federal Statistical System -- 1958

1. General Coordination: Office of Statistical Standards
   (Bureau of the Budget)

2. General-Purpose Statistical Agencies

   Statistical Branches of Agricultural Marketing Service
   (Department of Agriculture)

   Bureau of Labor Statistics
   (Department of Labor)

   Bureau of the Census
   (Department of Commerce)

   National Office of Vital Statistics
   (Department of Health, Education, Welfare)

3. Analytic and Research Agencies

   Council of Economic Advisers (Executive Office of the President)

   Office of Defense Mobilization (Executive Office of the President)

   Division of Agricultural Economics (Department of Agriculture)

   Office of Business Economics (Department of Commerce)

   Division of Research and Statistics (Federal Reserve System)

   Production Economics Research Branch (Department of Agriculture)

   Business and Defense Services Adm. (Department of Commerce)

   Bureau of Foreign Commerce (Department of Commerce)

   Bureau of Mines (Department of Interior)

4. Other Administrative, Regulatory, and Defense Agencies

A Study of Data by a Table and a Graph

So many decisions of Uncle Sam depend on the number of people in the country that we should first examine the population in various years: *

Table I
Population Facts about the United States

Year	Population in Millions	Increase	Per cent of Increase
1790	3.9		
1800	5.3	1.4	35.1
1810	7.2	1.9	36.4
1820	9.6	2.4	33.1
1830	12.9	3.3	33.5
1840	17.1	4.2	32.7
1850	23.2	6.1	35.9
1860	31.4	8.2	35.6
1870	39.8	8.4	26.6
1880	50.2	10.4	26.0
1890	62.9	12.7	25.5
1900	76.0	13.1	20.7
1910	92.0	16.0	21.0
1920	105.7	13.7	14.9
1930	122.8	17.1	16.1
1940	131.7	8.9	7.2
1950	150.7	19.0	14.5

---

* From STATISTICAL ABSTRACT OF THE UNITED STATES, 1956

Exercises

A careful study of such a statistical table will help you to answer the questions in Exercises 1 through 8.

1. Do you see any general trends in the table?
2. Do any particular figures seem out of line? Why do they seem so?
3. In which decade was the percent of increase the highest? From your study of history do you know the reason?
4. In which decade was the per cent of increase the lowest? Can you explain it by any history of that period that you have studied?
5. In which decade did the second lowest per cent of increase occur? Why?
6. When was the "Irish Famine"? How did that affect the population of the United States?
7. What was the increase in population from

   (a) 1800 to 1850? (b) 1850 to 1900? (c) 1900 to 1950?
8. What was the per cent of increase in each of the periods mentioned in Exercise 7?
9. Construct a broken line graph of the population of the U. S. A. at the end of each decade as given in the table on page 5. Then answer the following questions:
   (a) What general trend does the graph show?
   (b) In which 50-year period did the population grow the fastest?
   (c) If the population increases at the same rate from 1950 to 1960 as in the previous decade (if the graph climbs in a straight line from 1940 to 1960) what will be the population in 1960?
   (d) Why does Uncle Sam need to know, at least approximately, how many people there will be in the U. S. next year? In 1960?

The following table shows the immigration to this country in recent years.

Table II
Immigration Facts about the United States

Period	Number of Immigrants	Millions
1820 - 1830	151,824	.2
1831 - 1840	599,125	.6
1841 - 1850	1,713,251	1.7
1851 - 1860	2,598,214	2.6
1861 - 1870	2,314,824	2.3
1871 - 1880	2,812,191	2.8
1881 - 1890	5,246,613	5.2
1891 - 1900	3,687,564	3.7
1901 - 1910	8,795,386	8.8
1911 - 1920	5,735,811	5.7
1921 - 1930	4,107,209	4.1
1931 - 1940	528,431	.5
1941 - 1950	1,035,039	1.0

Exercises

Use Table II to answer the following:

1. Do you see any general trends in certain years?
2. Which figures seem out of line in these trends. Explain why.
3. Examine this table along with the one on population and see if there are any similarities.
4. If you have not done Exercise 9, page 6, prepare a broken line graph showing the increase in population in each decade from 1790 to 1950. (Table I)

5. On the same page, draw a broken line graph showing immigration by decades from the above table.
6. For which periods was the increase in immigration also a period of increases in population?
7. What was the general trend in immigration
    (a) From 1830 to 1890? Why?
    (b) From 1890 to 1900? Why?
    (c) From 1900 to 1920? Why?
    (d) From 1920 to 1940? Why?
8. In how many periods was there an increase in immigration? In how many periods was there a decrease?
9. There were new"quota laws" in 1923. How did they affect immigration?
10. Describe the trend in immigration if it continues in the same way as in the last 20 years. Can you make an approximate prediction for the period 1950-60? Do you see why Uncle Sam needs to know such numbers in order to help get a correct estimate of population?

The Arithmetic Mean

You have now examined some of the statistics collected by Uncle Sam, you studied the tables, you constructed graphs for a closer study, and you saw how some predictions might be made on the basis of your study.

Statistics can be studied in another way, that is by using mathematics to a greater extent in studying the sets of data.

It is difficult, and often unnecessary, to have a clear mental picture of a set of data. It is helpful to try to describe the most important features by a few numbers. The most useful numbers of this kind are the arithmetic mean (sometimes called simply the mean), the median, the mode, the range, and the average deviation.

You have probably averaged several numerical grades in some classes by adding them and dividing by the number of grades. This is a useful kind of average and is called the arithmetic mean or the mean. We will use it to study a set of statistics collected by the federal government in a recent year.

Table III
Number of Unemployed Persons in the U. S.

Year	(a) Number of Thousands	(b) Number of Hundred Thousands	(c) Number of Millions
1932	12,060	121	12
1934	11,340	113	11
1936	9,030	90	9
1938	10,390	104	10
1940	8,120	81	8
1942	2,660	27	3
1944	670	7	1
1946	2,270	23	2
1948	2,064	21	2
1950	3,142	31	3
1952	1,673	17	2
1954	3,230	32	3
1956	2,834	28	3

One of the first and easiest facts to find in a set of data is the range, or the difference between the highest and lowest numbers. In the previous table, we see that the range (in numbers of millions) is 12 - 1 or 11.

To find an arithmetic average or mean, find the total of the set of numbers and divide by the number of items:

The total of the numbers in column (c) is 69.

The number of items is 13.

$$\frac{\text{Total}}{\text{Number of items}} = \frac{69}{13} = 5.3 \text{ (to the nearest tenth)}$$

Notice that the mean does not tell you the number for any one year, but if the unemployment were 5.3 millions each year for 13 years, the total would be 13 x 5.3 millions or 69 millions.

Did any particular year have the "average" number of unemployed persons?

Averages are often called "measures of central tendency". Do you see why?

Exercises

1. Find the mean of the following set of test scores:

   88, 78, 92, 85, 83, 90, 88, 85, 86, 91

2. From Table III find the mean of the number of unemployed persons, in number of thousands, column (a) from 1932 through 1940.

3. From Table III find the mean of the number of unemployed persons, in number of thousands, column (a) from 1942 through 1956.

4. The following temperatures were the noon readings for a week. Find the mean noon temperature for the week:

   52°, 48°, 65°, 62°, 67°, 70°, 56°

5. Why was unemployment so high in 1932? 1934?

6. Why was it low in 1942? 1944?

7. How does the amount of unemployment affect
   - (a) the number of automobiles that are sold?
   - (b) the amount of unemployment compensation to be paid?
   - (c) the amount of food and clothing that is bought?

The Median and the Mode

In a set of data, it is often useful to know what would be the middle number if they were arranged from smallest to largest. In such a case, half of the numbers would be larger and half would be smaller than the middle number which is called the median.

For an example, let us examine the heights of pupils in a certain class of 13 pupils.

Pupil	Inches in Height
1	53
2	57
3	61
4	62
5	51
6	61
7	54
8	59
9	61
10	60
11	65
12	52
13	53

To find the pupil in the group, so that half the group is taller and half the group is shorter, picture the pupils arranged in order of height. The measurements, arranged in order, would be:

51, 52, 53, 53, 54, 57, 59, 60, 61, 61, 61, 62, 65

By counting from the left or right, the middle of the set of 13 items would be the 7th one. In this case, it is 59, so we say that the median of this set of numbers is 59.

If the largest number in this set of figures happened to be 70 instead of 65, would that affect the median? Why?

If the lowest numbers were 41 and 42 instead of 51 and 52, would that change the median?

Let us compare the mean or arithmetic average of this set of figures with the median. The sum of the numbers is 749.

$\frac{749}{13} = 57.6$. Thus the mean is 57.6 . How does this compare with the median?

If the largest number in this set of figures were 80 instead of 65, would that affect the mean? Why?

Notice that even though the median and mean are called "averages", they may not be the same number. You will find that each is useful for particular kinds of cases.

Exercises

1. Find the median and the mean of this set of scores on an arithmetic test: 83, 81, 76, 94, 87, 71, 90, 88, 73, 95, 67, 86, 87, 89, 93
2. Make a table for the heights and weights of the pupils in your class. Calculate
   (a) the average height (mean)
   (b) the median height
   (c) the average weight (mean)
   (d) the median weight
3. Find the median temperature of the data in Exercise 4, page 10.

Grouping Data

If you were listing heights of a very large group of pupils, you would not be able to list each one separately. You might group the figures in some such way as this:

Height in inches	Number of pupils
46 - 49	14
50 - 52	33
53 - 55	57
56 - 59	42
60 - 63	17
64 - 67	12

In order to find the middle pupil, find the total number and divide by 2. 14 + 33 + 57 + 42 + 17 + 12 = 175 . $\frac{175}{2} = 87\frac{1}{2}$. So the middle person will be the 88th one, counting from the top or bottom. If we count down from the top, 14 + 33 = 47. We need 41 more to reach 88. Counting down 41 more in the group of 57 brings us to the lower part of that group. Since the 88th person is within that group, we say that the median height of the whole group of pupils is between 53 and 55 inches. Since the 88th person comes rather low in that group as we count down, we say that the median height is likely to be nearer 55 than 53.

Let's check our work and count up from the bottom to the 88th person. 12 + 17 = 29. 42 more makes 71. So the 88th person is not in the lowest three groups. We need 17 more than 71 to make 88, so we count 17 more and that takes us into the group of 57 as we found when we counted down from the top. Again you find the 88th person in the group of 57 whose height is between 53 and 55. Thus the median height of the group is between 53 and 55 inches.

An important set of information which Uncle Sam needs each year is the data on incomes, since over 50% of the national income comes from individual income tax.

Following are the data for a recent year.

Table IV

Income	Number of Families (Thousands)
Under $1000	3,227
1000 - 2000	6,022
2000 - 3000	7,164
3000 - 4000	8,192
4000 - 5000	7,455
5000 - 6000	5,580
6000 - 7500	5,323
7500 - 10,000	3,390
10,000 - 15,000	1,899
15,000 - 20,000	523
20,000 - 25,000	274
25,000 - 50,000	336
Over 50,000	95

In a set of statistics, the value that occurs most often is called the mode.

Which income was earned by more people than any other? In other words, which is the mode of this set of figures?

You can find the mean of such a set of figures if you can find the total amount earned in each bracket. But do you know how much each of the 95,000 people earned in the "over 50,000" bracket? Might some have earned $500,000? Or $1,000,000? Or more? It is not possible to find the mean of this set of figures since we do not know how large the incomes are in the last group. Such a set of figures is called "open-ended".

It is possible, however, to find the median, or the salary bracket of the middle family. Then we will know that half of the families earned more than this amount and half earned less.

First find the total number of families and divide by two to find the middle number. Then count down from the top figure or up from the bottom figure, as taught on page 13, until you reach that number.

Exercises

1. Make a bar graph of the data in Table IV.
2. How can you identify the mode on the graph?
3. Estimate, if possible, the median salary by examining the graph carefully.
4. Find the mode in the following list of test scores: (Do not group.)

   85, 79, 82, 93, 85, 78, 82, 91, 85, 74, 93, 86, 90, 85, 78, 81, 85, 86, 94, 85, 84, 68, 83, 91, 82
5. Find the median score in the data, Exercise 4.
6. Find the median by grouping the following data on temperatures: (use intervals of 5, like 70-74, 75-79, and so on.)

   62°, 74°, 73°, 91°, 68°, 84°, 75°, 76°, 80°, 77°, 68°, 54°, 68°, 72°, 71°, 86°, 82°, 74°, 55°, 72°, 50°, 63°, 71°.

Table V
Patents Issued, 1901 - 1955

Period	Total	Number of Thousands	Total since 1900 (Thousands)
1901 - 1905	148,291	148	148
1906 - 1910	175,618	176	324
1911 - 1915	194,387	194	518
1916 - 1920	207,108	207	725
1921 - 1925	217,525	218	943
1926 - 1930	234,857	235	1178
1931 - 1935	256,219	256	1434
1936 - 1940	229,514	230	1664
1941 - 1945	184,573	185	1849
1946 - 1950	163,122	163	2012
1951 - 1955	209,215	209	2221

7. Draw a vertical bar graph of the number of thousands of patents issued by five year periods since 1900 (Table V).

8. Draw a broken line graph showing the total number of patents by five year periods since 1900.

9. Do you see any trends in the graphs of Exercises 7 and 8? What are they?

The Mean (or Average) Deviation

Another useful method of studying a set of numbers is to find the deviations or difference of each number from the mean.

Consider the set of numbers: 1, 2, 5, 5, 6, 7, 9.

The mean $= \frac{1 + 2 + 5 + 5 + 6 + 7 + 9}{7} = \frac{35}{7} = 5$

The deviations (differences) from the mean are:

5 - 1, 5 - 2, 5 - 5, etc., or 4, 3, 0, 0, 1, 2, 4.

The average of these diviations is:

$$\frac{4 + 3 + 0 + 0 + 1 + 2 + 4}{7} = \frac{14}{7} = 2$$

Let us see how the above measure helps to throw light on a set of data.

The total receipts of the federal government in the years 1946 - 1955 were as follows:

Year	Billions	Deviations from Mean
1946	44	- 11.5
1947	45	- 10.5
1948	46	- 9.5
1949	43	- 12.5
1950	41	- 14.5
1951	53	- 2.5
1952	68	+ 13.5
1953	73	+ 18.5
1954	73	+ 18.5
1955	69	+ 14.5
	555	126

The arithmetic mean of these receipts is the total, 555, divided by 10, or 55.5.

The second column shows the differences of each year's receipts from the mean, 55.5. The - 11.5 means that 44 is 11.5 below the mean, for instance, and + 14.5 means that 69 is 14.5 above the mean. Have you seen minus signs used in a similar way to those in this table for temperature? Any other place?

Exercises

1. In which year was the deviation from the mean the greatest? Can you think of reasons for this?
2. In which year was the deviation from the mean the least?
3. Find the average deviation by finding the total of the deviations and dividing by 10. The signs before the numbers are disregarded

since we want to know the *size* of the deviations, no matter which direction they are from the mean.

4. Find, to the nearest tenth, the mean and average deviation of the following test scores:

   85, 82, 88, 76, 90, 84, 80, 82, 84, 83

5. Find the mean and average deviation of the test scores (same test, but in another class):

   94, 84, 68, 74, 98, 70, 96, 84, 76, 96

6. (a) Compare the means of Exercises 4 and 5.

   (b) Compare the average deviations of Exercises 4 and 5.

   (c) What information does the average deviation tell about the data which cannot be found by just using the mean?

Summary Exercises

Table VI
Gross Debt of the United States for the Years 1861 to 1955
(Five-year Periods)

(a) Years	(b) Millions	(c) Billions	(d) Deviations from Mean
1861 - 65	2,678	3	
1866 - 70	2,436	2	
1871 - 75	2,156	2	
1876 - 80	2,091	2	
1881 - 85	1,579	2	
1886 - 90	1,122	1	
1891 - 95	1,097	1	
1896 - 1900	1,263	1	
1901 - 05	1,132	1	
1906 - 10	1,147	1	
1911 - 15	1,191	1	
1916 - 20	24,299	24	
1921 - 25	20,516	21	
1926 - 30	16,185	16	
1931 - 35	28,701	29	
1936 - 40	42,968	43	
1941 - 45	258,682	259	
1946 - 50	257,357	257	
1951 - 55	274,374	274	

1. (a) What is the mode of the set of figures in column (c), Table VI? In other words, which gross debt appears most often?

   (b) Does this number tell you very much about the total debt picture of the United States Treasury since 1861?

2. (a) Find the median of the set of figures in column (c) which will give you the median national debt from 1861 to 1955. This means that you will need to find the middle figure between 1 and 274. In order to do this, list the numbers from column (c) in order, and find the middle one on the list by counting up from the bottom or down from the top.

   (b) Does the median tell you much about the total debt over this period of years?

   Do you see that it is necessary to find the mean in order to take into consideration the very large numbers in the table?

3. Find the mean of the set of numbers in column (c) to the nearest whole number.

4. Find the deviations from the mean and then the average deviation to the nearest whole number.

5. Explain the large deviations both below and above the mean.

6. The following information is on salaries:

   $4,000, $6,000, $12,500, $5,000, $5,000, $7,000, $5,500, $4,500, $5,000, $6,500, $5,000

   (a) Find the mean of the data.

   (b) How many salaries are above the mean?

   (c) How many salaries are below the mean?

   (d) Does the mean seem to be a fair way to describe the "average" of this data?

   (e) Find the median of the set of data.

   (f) Does the median seem to be a fair way to describe the "average" of this data?

7. Repeat the parts of Exercise 6 using the following data on temperatures:

   47°, 68°, 58°, 80°, 42°, 43°, 68°, 74°, 43°, 46°, 48°, 76°, 48°

****

From the illustrations given, you have studied how Uncle Sam needs mathematics to collect and study statistics.

You have seen how mathematics is used to interpret these statistics and help the government in making decisions that affect you and me and every citizen.

The next time that you see a set of figures in the newspaper, look them over carefully, try to recognize trends, and consider whether decisions based on them, and affecting you, will be made.

UNIT XIII

CHANCE

Recently the picnic of the employees of a weather bureau was called off on account of rain. Even an expert weather forecaster cannot say for sure what tomorrow's weather will be.

If you plan a picnic you may first read the weather report for the day of the picnic. This report may read "Probable showers" or "Clear and warm." Sometimes the weatherman is correct and sometimes he is wrong. His forecast is an example of his best available estimate of what will happen. His forecast is a "Chance Statement". He, himself, does not know with complete certainty whether it is true or false. In this unit we shall study some ideas about chance statements like, "The Yankees will probably win," or "There is a 50-50 chance that a head will show if a penny is tossed and allowed to fall freely."

The table below shows a number of weather forecasts from April 1 to April 11 and, also, the actual weather.

Table

Date	Forecast	Actual weather	Truth Value of the Forecast
1.	Rain	Rain	True
2.	Light showers	Sunny	False
3.	Cloudy	Cloudy	True

Date	Forecast	Actual weather	Truth Value of the Forecast
4.	Clear	Clear	True
5.	Scattered showers	Warm and sunny	False
6.	Scattered showers	Scattered showers	True
7.	Windy and cloudy	Overcast and windy	True
8.	Thundershowers	Thundershowers	True
9.	Clear	Cloudy and rain	False
10.	Clear	Clear	True

Each forecast is an example of a "statement". A statement is a collection of words or symbols which we can say is either true or false (but naturally not both). In the case of the first forecast the statement was found to be true after observing the weather. Also, observations show that statements 3, 4, 6, 7, 8, and 10 were true. However, statements 2, 5, and 9 proved false since the observed weather was different from the forecast. Hereafter we shall write "true" as "T" and "false" as "F". The T's and F's are called "Truth values of the statement."

In addition to these symbols it is useful in dealing with chance statements to have a number which measures the chance that a statement is true. The weather man was correct 7 out of 10 times. Written as a ratio this is $\frac{7}{10}$. This $\frac{7}{10}$

is used as the best estimate which we can make from the given information of the measure of the chance that the next weather forecast will be true.

In this case, since we had such a small number of forecasts, it would not be fair to say that the weatherman will be correct $\frac{7}{10}$ of the time in future forecasts. To make such a conclusion requires an understanding of much more mathematics that we can study in this unit. It does, however, give us some clue as to the chance that the weatherman may be correct.

When we find the measure of the chance that statements will be true, we shall say that we are estimating the true measure from the best information we have. This might be called estimating the true value of the chance statement.

Written as a formula this is:

The measure of the chance that a statement is true =

$$\frac{\text{The number of times the statement is T}}{\text{The total number of times the statement is T or F}}$$

or

$C = \frac{T}{S}$, where T means the number of times the statement is true, S means the sum of the number of T's and F's, and C is the measure of the chance.

Exercises

1. The following statement is made:

"The stoplight at Main Street and Center Street will be green whenever I am arriving at the intersection".

a. Why is this a "statement"?

b. Below is a table of the actual color when I was arriving at the intersection. Copy the table and in the column at the right, write whether the statement above is T or F. You may use "T" for true and "F" for false if you like.

Table

The light	Truth value of the Statement
1. Green	
2. Red	
3. Amber	
4. Green	
5. Red	
6. Amber	
7. Red	
8. Green	
9. Green	
10. Amber	
11. Red	
12. Green	
13. Red	
14. Amber	
15. Green	
16. Red	

c. Write a number or measure for the chance that the statement is correct.

2. Change the statement in Ex. 1 to read "Red" instead of "Green". Use the same observations as in part b and find a number or measure for the chance that the new statement is correct.

3. Given: 3 red marbles and 2 white marbles are in a box. You are to take out of the box, without looking inside it, two marbles. What is the chance that both marbles will be red? (Sometimes we merely say "What is the chance?" when we mean "What is the number which measures the chance that the statement is true?")

In this problem the chance statement is: In a drawing, two red marbles will be drawn from a box containing 3 red and 2 white marbles.

The following table lists the possible pairs which may be drawn. We assume that each red marble has the same likelihood of being drawn. Let's call the three red marbles R_1, R_2, and R_3 and similarly the white marbles shall be called W_1, and W_2. Complete the table of truth values and find the number for the chance that the statement is correct. Mark as T those draws which consist of 2 red marbles.

Draws	Truth Value
1. $R_1 R_2$	
2. $R_1 R_3$	

Draws	Truth Value
3. $R_1\ W_1$	
4. $R_1\ W_2$	
5. $R_2\ W_1$	
6. $R_2\ W_2$	
7. $R_2\ R_3$	
8. $R_3\ W_1$	
9. $R_3\ W_2$	
10. $W_1\ W_2$	

4. Two black marbles and one white marble are in a box. You are to take out, without looking inside the box, two marbles. What is the chance that one marble is black and the other is white?

Suggestions: Make a table as in Ex. 3. List all possible pairs and find the truth value of each pair. You should have three pairs.

5. In Ex. 4, what is the chance that both marbles drawn are black?

6. Three hats are in a dark closet. Two belong to you and the other to your friend. Being a polite person, when your friend is ready to leave with you, you reach in the closet and draw any two hats. What is the chance that each of you gets his own hat?

7. A committee of two is selected from a group of 4 people. You are one of the 4. If the person selecting the committee

has no favorites and, therefore, selects without a plan, what is the chance that you will be on the committee?

8. In Ex. 7, your favorite friend is also one of the 4 people. What is the chance that both you and your friend will be selected?

Models for Chance Statements

You may have noticed two kinds of problems in the set of exercises above. One kind uses observations to determine whether a statement is T or F. (Exs 1, 2, and the example page 1). However, in the marble problems and the hat problem experimental observations were not used. All possible events were listed and the measure of chance was based upon these possibilities. Thus, in these cases a "model" was made to represent the problem. It is also necessary to point out that each event in a model, say the 10 pairs in Ex. 3, has the same chance of occurring. In the case of Ex. 3, each pair has a $\frac{1}{10}$ chance of occurring.

In the exercises above finding an estimation or measure of the chance that a statement is true was a matter of counting. First we were talking about a statement and, secondly, we were able to count all possible (or available) T's and F's. Although more advanced mathematical processes have been set up to find measures of chance where counting is impossible, counting is essential in many situations.

An easy type of model to talk about is tossing a penny or pennies. A penny has a head and a tail (let's assume here that the coin cannot stand on edge if it is tossed and allowed to fall freely on a flat surface). Also, in all problems about coins, we assume that the coin is balanced (honest). We then expect heads to appear about as many times as tails if the coin is tossed a great number of times.

Example 1: Find the measure of the chance that the statement, "If two pennies are tossed, exactly one head shows," is true.

Solution:

a. Make a table showing all possible events.

1st penny	2nd penny	Truth value
Head	Head	F
Head	Tail	T
Tail	Head	T
Tail	Tail	F

Note: In this model, each event (each possible result of the tossing of two coins), one of which is, for example, a head and a tail, has the same chance of occurring; that is, $\frac{1}{4}$.

b. There are 4 possible ways the pennies may fall. In the table above the truth values of each event has been written.

If both pennies are heads, the statement above is F; if the first penny is a head and the second is a tail, the above statement is T; and so on.

c. Count the number of times the statement is true. This number is the numerator of measure of chance. The sum of the number of T's and F's is the denominator.

The chance that exactly one head will show if two coins are tossed is $\frac{2}{4}$ or $\frac{1}{2}$ or $c = \frac{1}{2}$ in this case.

Example 2: Find the measure of the chance that the statement, "If three pennies are tossed, exactly two heads show," is true.

Solution:

a. Make a table of all possible ways the 3 coins may fall and after each event write T if the event makes the statement true and F if the event makes the statement false.

1st penny	2nd penny	3rd penny	Truth value
Head	Head	Head	F
Head	Head	Tail	T
Head	Tail	Head	
Head	Tail	Tail	
Tail	Head	Head	
Tail	Head	Tail	
Tail	Tail	Head	
Tail	Tail	Tail	

(Complete the table)

b. Count the T's for the numerator and both the T's and the F's for the denominator. Then, if 3 pennies are tossed the chance that exactly two heads show is $\frac{3}{8}$ or $c \overset{m}{=} \frac{3}{8}$.

Exercises

For each exercise find the measure of the chance statement.

1. If two pennies are tossed, exactly two heads show.
2. If two coins are tossed, at least one head shows.
3. If two coins are tossed, exactly one tail shows.
4. If three coins are tossed, exactly one head shows.
5. If three coins are tossed, exactly three heads show.
6. If three coin are tossed, at least one head shows.
7. In example 1, the table shows four events.
   a. Find the measure of the chance statement for each event.
   b. Find the sum of the four measures of chance.
8. Do the same for example 2.
9. If four coins are tossed, what is the chance of exactly 3 heads showing? Of exactly 1 tail showing? Of exactly two heads showing? Of exactly four heads showing?

10*. Three cards are numbered like the ones below.

$\boxed{3}$ $\boxed{5}$ $\boxed{6}$

Without looking at the numbers you are to draw any two cards. What is the chance that the sum of the two numbers is odd?

11*. You are to be placed in a line with two girls (or boys), one of whom is your favorite. What chance is there that you will stand next to your favorite? In such a problem we assume that you are not placed according to any plan (including your own). If you crowd in next to your favorite, chance would not play a role.

Decisions and Chance

Suppose a person must make a choice between two possibilities. Should he just chose one by guessing or is there a better method for him to try? A person may make a wiser decision by observing and comparing statements about the chance of the two events in question.

Example: Which of the two following statements is more likely to be true?

A. If 3 pennies are tossed, exactly two heads appear.

B. If 2 pennies are tossed, exactly one head appears.

Solution: A

The measure of the chance of A = $\frac{3}{8}$

The measure of the chance of B = $\frac{1}{2}$

Conclusion: Since $\frac{1}{2}$ is greater than $\frac{3}{8}$, statement B is more likely to be true.

In order to solve a problem of this kind, find for each statement the measure of the chance that it will be true and select the statement corresponding to the larger measure.

Exercises

For each exercise find a measure of the chance statement for each event and choose, if possible, the event most likely to happen.

1. Statement A: If three coins are tossed, exactly 2 tails show.

Statement B: If two coins are tossed, exactly 1 head shows.

2. A: If 3 coins are tossed, at least 2 heads show.

B: If 2 coins are tossed, exactly 1 tail shows.

3. A: If 4 coins are tossed, exactly 3 heads show.

B: If 3 coins are tossed, exactly 2 heads show.

4. A: It rains on Friday the 13th.

B: The sun shines on Friday the 13th.

The following table shows the weather on 20 Friday the 13ths. From the table decide which is more likely to happen over a great number of Friday, the 13th's.

	Truth Value of Statement A	Truth Value of Statement B
1. Heavy rain		
2. Light rain		
3. Sunny		
4. Usually sunny, no rain.		
5. Sunny		
6. Scattered showers		
7. Showers		
8. Sunny		
9. Sunny		

	Truth Value of Statement A	Truth Value of Statement B
10. Sunny		
11. Cloudy, no rain		
12. Partly cloudy		
13. Cloudy with some showers		
14. Showers		
15. Sunny		
16. Sunny		
17. Hot and sunny		
18. Sunny		
19. Cloudy and some showers		
20. Sunny		

Experiments

1. a. If one penny is tossed what is the chance that a head shows?

   b. How many heads would you expect to get if the penny is tossed 50 times?

   c. Toss a penny 50 times and compare the observed results with the measure found in part b. You are extremely lucky if they agree exactly.

2. Keep a record (as in the Example on page 1) of weather forecasts of a morning paper and an evening paper. Find chance statements for each paper and compare their measures.

3. In Ex. 4, page 5, the chance was estimated by making a "model" of the possible events.

The solution of this problem could be approached by carrying on an experiment. For example, repeat the drawing of a pair 50 times (after each drawing replace the 2 marbles that you have drawn). Each draw then makes the statement T or F. Apply our method for calculating a measure of a chance statement.

Compare this measure to the measure obtained from the model. It is not very likely that the two measures each found a different way, will be exactly alike. In a further study of chance you will study more about how close you could expect the two measures to be.

4. Set up an experiment to answer the question: What chance is there that exactly two heads will show if three coins are tossed?

5. Set up an experiment to answer the question. What chance is there that exactly 5 heads will show if ten coins are tossed?

6. a. Mark off on heavy cardboard parallel lines, two inches apart.

   b. Make ten sticks from wooden matches (or tooth picks) 1 inch long.

   c. Hold the ten sticks about 8 or 9 inches above the cardboard and drop them.

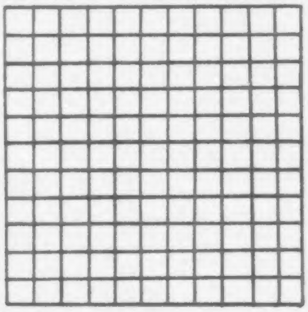
d. Count the number of sticks which touch or cross a line of the cardboard.

e. Repeat the experiment 50 times, each time counting and recording the number of sticks which touch or cross a line.

f. Find the average number of sticks that touch or cross a line out of the 10 that are dropped.

7. You are to decide whether or not commercial fertilizer helps seeds germinate and grow rapidly.

Set up the following experiment to help you make a decision: Bring two boxes about a foot square and 6 inches deep. Fill each with soil from the same area. With the help of your science teacher make sure that the moisture in the two boxes of soil is about equal. Then add commercial fertilizer to one of the boxes. Make a grid with strings like the sketch below for each box.



Two sets of 10 strings. The strings of each set are 1 inch apart.

At each intersection, plant corn seeds. There will be 100 seeds planted in each box. Add the same amount of water to each box and place both boxes in the same part of the room. Consult your science teacher in regard to the amount of water to be added. The seeds should all be planted the same depth in the soil.

a. Decide how you would use chance to help decide whether or not fertilizer helped the seeds germinate.

b. Decide how you would further your experiment to decide whether or not the plants grew more rapidly under the conditions of commercial fertilization.

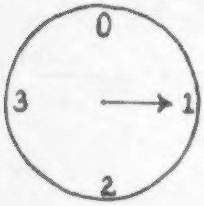
8. Some members of the class may carry on the experiment described in Ex. 7 but first they should test the soil to determine the best kind and amount of fertilizer to use. The science teacher may be able to help with this, the county agent may have soil testing kits to lend, sell, or give away. or, finally, there may be a nearby soil testing station to which a sample may be taken.

9. Repeat the type of experiment described in Ex. 7 but put fertilizer in both boxes. However, add twice as much in one and decide if the extra fertilizer produced better results.

UNIT XIV

MATHEMATICAL SYSTEMS

A New Arithmetic



$1 + 2 = 3$

$2 + 3 = 1$

$2 + 2 = 0$

Here are some new number facts. Do they seem a little strange to you? We might call this "clock arithmetic." We have used a four-minute clock -- one which might be used to time rounds and intermissions in a boxing match.

Let's see how this kind of arithmetic works. If the hand is at 1 minute, and moves for 2 minutes, then it is at 3. We can write $1 + 2 = 3$. If it is at 2 and moves for 2 minutes then it is at 0. We write $2 + 2 = 0$. If it is at 2 and moves for 3 minutes, then it stops at 1. We write $2 + 3 = 1$.

We can make an addition table for this system of arithmetic thus:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

We read tables of this sort by following across horizontally from any entry in the left column, say 2, to the position below some entry in the top row, say 3. The entry in this position in the table(s) is then taken as the result of combining the element in the top row with the element originally picked out in the left hand column. In the case above we write $2 + 3 = 1$. Check that $3 + 1 = 0$ in this table. Studying our table, is $1 + 2 = 2 + 1$? is $2 + 3 = 3 + 2$? What does this suggest to us about this kind of arithmetic?

Is (1 + 2)+ 3 = 1 +(2 + 3)?

Is 1 + (1 + 2) = (1 + 1) + 2?

Check some other examples. What does this suggest to us?

Let's compare this new arithmetic with ordinary arithmetic.

1. What are the numbers of the ordinary arithmetic? Of this new arithmetic?
2. Does addition have the commutative property in this new arithmetic?
3. Does addition have the associative property in this new arithmetic?
4. Is there an identity element (an element which when combined with any other element produces the "other" element itself as the result) for addition in this new arithmetic?

We call this new kind of arithmetic "modular arithmetic", and the number 4 is called the modulus. We say this system is arithmetic mod 4. The arithmetic of the three-minute egg timer is arithmetic mod 3. We can write an addition table for mod 3, mod 5, mod 8 -- we can have as many modular arithmetics as we have natural numbers.

Exercises - 1

1. Make an addition table for mod 3, mod 5, mod 6, and mod 8. What are the numbers for each?
2. Using the mod 5 addition table, find simpler names for:

1 + 2
3 - 1
0 - 3
4 + 1
3 + 2
3 + 4
1 - 2
3 - 2

2 - 4
4 + 0
(3 + 2) + 4
3 + (2 + 4)
(4 - 2) + 3
(3 - 1) + (2 + 4)
4 - (2 - 3)

(3 + 4) - 3
(2 - 3) - 3
(1 - 4) + 4
3 + (3 - 1)
(3 + 2) + (2 - 1) - (4 - 3)

3. You have only a five-minute clock. How many times would the hand go around if you were using it to tell you when 23 minutes had passed? Where would the hand be at the end of the 23 minute interval? If you continue then for 15 minutes where is the hand?

   Can you figure out an easy way to work problems like this without counting on the clock? Try it.

What is an Operation?

We are familiar with the operations of ordinary arithmetic -- addition, multiplication, subtraction, and division. In the preceding exercises we did a different kind of operation. We made a table for the new addition. This operation is defined by the table, because it tells us what we get when we put two numbers together. Study the following tables.

(a)

+	1	2	3	4	5
1	2	3	4	5	1
2	3	4	5	1	2
3	4	5	1	2	3
4	5	1	2	3	4
5	1	2	3	4	5

(b)

+	3	5	7	9
3	6	8	10	12
5	8	10	12	14
7	10	12	14	16
9	12	14	16	18

(c)

+	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	1
2	3	4	5	6	7	8	9	10	11	12	1	2
3	4	5	6	7	8	9	10	11	12	1	2	3
4	5	6	7	8	9	10	11	12	1	2	3	4
5	6	7	8	9	10	11	12	1	2	3	4	5
6	7	8	9	10	11	12	1	2	3	4	5	6
7	8	9	10	11	12	1	2	3	4	5	6	7
8	9	10	11	12	1	2	3	4	5	6	7	8
9	10	11	12	1	2	3	4	5	6	7	8	9
10	11	12	1	2	3	4	5	6	7	8	9	10
11	12	1	2	3	4	5	6	7	8	9	10	11
12	1	2	3	4	5	6	7	8	9	10	11	12

(d)

▱	0	1	2	3
0	0	1	2	3
1	1	3	4	5
2	2	5	6	7
3	3	7	8	9

(e)

⊖	1	2	3
1	2	1	2
2	1	2	3
3	2	3	1

We see that these tables show us a way to put two things together to get one and only one thing. For example,

$2 + 2 = 4$ in both tables (a) and (c)

2 ▱ 1 = 5 and 2 ▱ 2 = 6 in table (d)

$P_2 * P_3 = P_5$ $\qquad$ $1 \ominus 1 = 3$

When we have a way of putting two things of a given set together to get a third, we say we have a binary operation. For instance,

8 and 2 when added gives us 10.

8 and 2 when multiplied gives us 16.

8 and 2 when "sumptified" gives us 18. When 6 and 4 are sumptified we get 16. 5 and 1 when sumptified give us 11. Were any of the tables a "sumptification" table?

This doesn't mean that we can always put things together in any order. For example,

2 ▱ 1 = 5, but 1 ▱ 2 = 4

For this reason, we must first remember that when we explained how to read a table we decided to write the element in the left hand column first and the element in the top row second with the operation's symbol between them. We must then remember to examine each new operation to see if it is commutative and associative.

Exercises - 2

1. Use the tables of operation on page 3 to answer these questions.

A	B
(a) $3 + 3 = ?$ if we use table (a)	(a) $12 + 12 =$
(b) $3 + 3 = ?$ if we use table (b)	(b) $10 + 6 =$
(c) 3 ▱ 2 =	(c) $3 \ominus 1 =$

(d) $2 \ominus 2 =$

(e) $1 \ominus 1 =$

(f) $11 + 12 =$

(d) 1 ▱ 3 =

(e) 1 ▱ 1 ▱ 2 =

(f) $2 \ominus 3 \ominus 3 =$

2. Which of the binary operations described in the tables on page 3 are commutative? associative? Is there an easy way to tell if an operation is commutative when you examine a table of operations? What is it?

3. Are the following binary operations commutative? associative?

   (a) Set: All natural numbers less than 50.

   Operation: Twice the first added to the second.

   Example: 3 combined with 5 produces 11 ($2 \times 3 + 5 = 11$)

   (b) Set: All natural numbers between 25 and 75.

   Operation: The first or lesser number.

   Example: If the two numbers are 28 and 36, the third number associated with them by this operation is 28.

   (c) Set: All natural numbers between 500 and 536.

   Operation: The second or greater number.

   Example: If the two numbers are 520 and 509, the third number is 520.

   (d) Set: The prime numbers

   Operation: The larger number.

   (e) Set: All natural numbers.

   Operation: Least Common Multiple.

   Example: If the numbers are 4 and 6, the third number determined by this binary operation is 12.

   (f) Set: All natural numbers.

   Operation: Greatest Common Factor.

   (g) Set: All natural numbers.

Operation: Given two natural numbers, m and n, the result of the operation is m^n.

(h) Set: The prime numbers.

Operation: The larger number.

(i) Set: Gallon cans of paint in different colors.

Operation: Mixing paint.

4. Make up a table for an operation that has the commutative property.
5. Make up a table for an operation which does not have the commutative property.

More about Closure

We already have an acquaintance with the idea of closure. What do you remember from that brief introduction?

We recall that a set is closed under an operation if we can always do that operation on any two members of the set and get a third number which is a member of the same set. The two members we start with may be the same one. For example,

(1) We observed that the set of even numbers is closed under addition. This means that if we add any two even numbers, we get a third even number.

$2 + 2 = 4$ (We used the same number.)

$14 + 6 = 20$

$44 + 86 = 130$

(2) We observed that the set of odd numbers is not closed under addition. This means that if we add two odd numbers we do not get a third odd number. For example, $3 + 5 = 8$. Is this a case where one example is enough to show that closure does not hold? We can actually give more examples. The sum of two odd numbers is always outside the set of odd numbers.

(3) We observed that the set of natural numbers is not closed under subtraction, that is, take a pair of numbers, 6 and 9, and subtract. 6 - 9 -- but no natural number has the name, "6 - 9." Yet, we can subtract 6 from 9 to give us the number, 9 - 6. The standard symbol for this number is "3".

(4) We observed that the set of natural numbers is not closed under division. It is true that $\frac{8}{2}$ is a natural number, but there is no natural number, $\frac{9}{2}$. What are some other illustrations of closure, that is, sets closed under an operation and sets not closed under an operation?

Exercises - 3

1. Study again the tables on page 3. Which sets are closed under the operation? Which sets are not closed under the operation? How do you know?
2. Which of the systems described below are closed?
   (a) The set of even numbers under addition.
   (b) The set of even numbers under multiplication.
   (c) The set of odd numbers under multiplication.
   (d) The set of odd numbers under addition.
   (e) The set of multiples of 5 under addition.
   (f) The set of multiples of 5 under subtraction.
   (g) The set of even numbers used in telling time under clock addition.
   (h) The set of odd numbers used in telling time under clock addition.
   (i) The set of numbers mod 7 under subtraction.
   (j) The set of natural numbers less than 50 under the operation where the third number is the smaller of the two numbers.

(k) The set of prime numbers under addition.

(l) The set of numbers whose numerals in base 4 end in '0' or '2' under addition.

(m) The set of numbers whose numerals in base 5 end in '3' under addition.

Identities

In our study of the number one in ordinary arithmetic, we observed that any number multiplied by 1 gave that same number, that is, the product of any number and 1 is the number, like,

$2 \times 1 = 2 \quad 3 \times 1 = 3 \quad 156 \times 1 = 156 \quad \frac{2}{3} \times 1 = \frac{2}{3}$ or, for any number in ordinary arithmetic, $n \cdot 1 = n$.

In our study of the number zero, we observed that the sum of 0 and any number in ordinary arithmetic gave the number, that is, $2 + 0 = 2 \quad 3 + 0 = 3 \quad 468 + 0 = 468 \quad \frac{8}{5} + 0 = \frac{8}{5}$ or for any number in ordinary arithmetic, $n + 0 = n$.

One is the identity for multiplication in ordinary arithmetic.

Zero is the identity for addition in ordinary arithmetic.

What is the identity for the arithmetic of the 4-minute clock? for our ordinary clock?

What tables of operation in Exercises 1 have identities? What is the identity for each?

Inverses

If we add two things and get the identity for addition, then we call them additive inverses of each other. For example, in the table

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

0 is the identity.

$2 + 2 = 0$
$3 + 1 = 0$
$1 + 3 = 0$

These pairs of numbers, 2 and 2, 3 and 1, 1 and 3, are said to be inverses of each other. Each element of the set has an inverse. The inverse of 0 is 0, the inverse of 1 is 3, the inverse of 2 is 2, and the inverse of 3 is 1.

Exercises - 4

1. Study tables on page 3.
    (a) Which tables have an identity and what is the identity?
    (b) Pick out inverses in these tables. Does each member of the set have an inverse?

Some Algebraic Systems

We have an algebraic system when the following statements are true:

1. There is a set of things -- these things need not be numbers.
2. There are one or more operations.
3. There are some properties concerning the operations and the sets of things -- such as the commutative property, the associative property, closure, identities, inverses.

Let's look at egg-timer arithmetic -- arithmetic mod 3.

+	1	2	0
1	2	0	1
2	0	1	2
0	1	2	0

(a) It has a set of things. These are numbers -- 0, 1, 2.

(b) It has the operation, +.

(c) The operation of + has the commutative property. Can you tell by the table? If so, how? We cam make some checks too. 1 + 2 = 0 and 2 + 1 = 0, so 1 + 2 = 2 + 1.

(d) There is an identity for the operation, + (the number 0).

(e) Every member of the set has an inverse for the operation +.

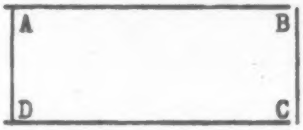
Algebraic Systems without Numbers

We may have algebraic systems without numbers in them. Suppose we invent one. What do we need?

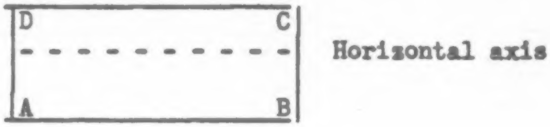
We must have a set of things. Then, we need some kind of operation -- something we can do with two of the things to get a third. And there must be properties concerning the operation and the things in the set.

Let's start with a post card -- really any rectangular shaped card will do. Instead of a set of numbers we will have a set of changes of position. We will take only those changes which make the card look like it did in the beginning (except that the marks on the corners may be moved around). How many of these changes are there?

We may start with it in some position which we will call the original position. We will say it looks like this:

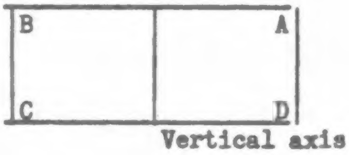


Letters in the corners of the card will help us see the different changes. One new position may be like this. The change of position was turning the card from its original position on its horisontal axis.

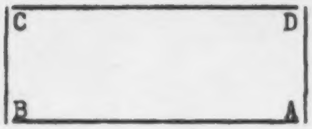


A second change of position is this:

(Turn the card from its original position on its vertical axis.)

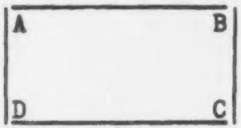


There is a third change -- we may turn the card from its original position halfway around its center. It looks like this:

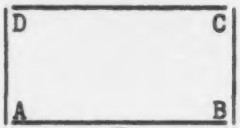


Another change is to leave the card alone.

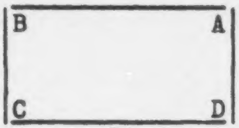
Here are the four changes of position:



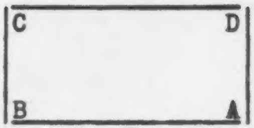
I



H



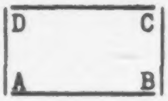
V



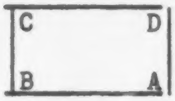
R

We can now make up our mathematical system. The set of things in our system is the set of changes I, H, V, and R. We will need an operation. Let's make up one -- it is *.

H * V means to first do change H



Then, do change V



This final position is the same change as change R. So, H * V = R

What shall we call this operation?

Complete this table:

*	I	H	V	R
I	I	H	V	__
H	H	__	R	__
V	__	__	I	H
R	__	__	H	I

Is this really an operation? What properties exist between the operation and set of things?

Exercises - 5

1. Examine the table of operations for the changes of the rectangle.

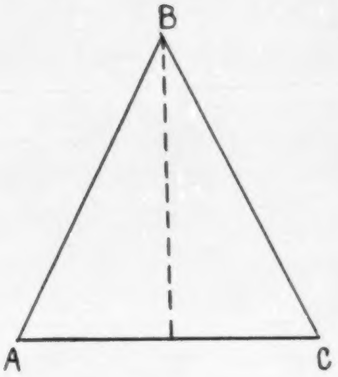
   (a) Is the set closed to this operation?

   (b) Is the operation commutative?

   (c) Is the operation associative?

   (d) Is there an identity for the operation?

2. Here is another system of changes. Take a triangle with two equal sides. Label the corners "A", "B", "C", so it will look like this:

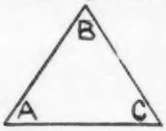


The set for the system will consist of two changes. The first change, called I, will be "leave alone". The second change, called M, will be "flip the triangle around its vertical axis."

$M^{\circ}I$ will mean flip the triangle about the vertical axis and then leave the triangle alone. How will the triangle look -- like it was left alone, I, or like the change M was done?

The operation, called $I^{\circ}M$ is: First do ______, and then do ______.

Always start the operation with the triangle in this position.



Does $M^{\circ}I$ M or does $M^{\circ}I$ I?

(a) Complete the table below:

o	I	M
I		
M		

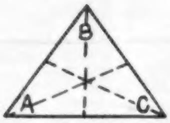
(b) Is the set closed for this operation?

(c) Is the operation commutative? associative?

(d) Is there an identity for the operation?

(e) Does each member of the set have an inverse for the operation?

3. Make a triangle with three equal sides. Label its corners "A", "B", "C", like this:



The set for this system will be made of six changes. Three of these will be flips about the axes, and three will be turning the triangle around its center.

Make a table for these changes. Examine the table. Is this operation commutative? Is there an identity change? Does each change have an inverse?

4. Try making a table of changes for a square.

There are eight changes. What are they? Is there an identity change? Is the operation commutative?

5. This is the table of changes of a triangle with two equal sides.

	I	M
I	I	M
M	M	I

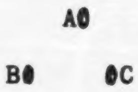
Fill in this table of operation for addition modulus two.

+	0	1
0		
1		

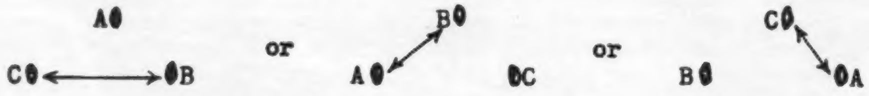
Suppose a "0" is put in place of every "I" in the table of symmetry, and a "1" is put in place of every "m" and a " " is put in place of the "0". What would the resulting table be?

The two tables use different symbols, but have the same pattern. We may then expect them both to have the same properties.

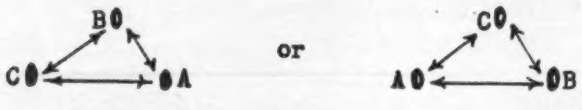
6. Another mathematical system which does not use numbers is the system of changing tires on a tricycle. Suppose tires on a tricycle are labeled like this:



By switching two tires we could get



By switching all three tires we could get



The set is made of tire switches, not tires.

(a) What will the operation be?

(b) Give a name to each of the five switches above. Let "I" be the "identity switch", that is the switch which makes no change at all. Make a table for the system of switching tires on a tricycle.

(c) Does this table have the same pattern as the table of changes for a triangle with three equal sides?

(d) Was the operation for changes of the triangle commutative?

(e) Is the operation of tire switching commutative? Check and see.

7. There are three pictures on the wall. We can leave them alone, switch two, or switch all three in various ways. See if you can make a system and a table for switching pictures. Remember to have a system you need a set of things, and an operation. What properties do you find in this system?

Algebraic systems may be defined without even having a geometric model. This can be done by merely giving the set of elements and the result of combining any two of them. Each of the following three tables defines an algebraic system.

(a)

π	R	W
R	R	W
W	R	W

(b)

*	P_0	P_1	P_2	P_3
P_0	P_2	P_3	P_0	P_1
P_1	P_3	P_2	P_1	P_0
P_2	P_0	P_1	P_2	P_3
P_3	P_1	P_0	P_3	P_2

(c)

~	△	▭	X	V
△	△	▭	X	V
▭	▭	X	V	△
X	X	V	△	▭
V	V	△	▭	X

Exercises - 6

1. Use these tables to complete the following statements correctly.

(a) $W \pi R =$

(b) $\triangle \sim X =$

(c) $V \sim V =$

(d) $R \pi W =$

(e) $P_1 * P_2 =$

(f) $P_2 * P_3 =$

(g) $P_0 * P_2 =$

(h) $\square \sim X =$

(i) $V \sim \square =$

(j) $W \pi W =$

(k) $R \pi R =$

(l) $P_3 * P_3 =$

2. Which one, or ones, of the binary operations π, $*$, $\sim$ has an identity element? What is it in each case?

3. Which one, or ones, of the operations π, $*$, $\sim$ is commutative? Prove your statement.

4. Use the tables to "compute" the following [Assume that parentheses, (), mean that the quantity enclosed by them is to be computed first and then treated as a single element]:

(a) $P_0 * (P_1 * P_2) =$ ____

(b) $(P_0 * P_1) * P_2 =$ ____

(c) $P_0 * (P_1 * P_3) =$ ____

(d) $(P_0 * P_1) * P_3 =$ ____

(e) $(P_2 * P_0) * P_3 =$ ____

(f) $P_2 * (P_0 * P_3) =$ ____

(g) $\triangle \sim (\triangle \sim X) =$ ____

(h) $(\triangle \sim \triangle) \sim X =$ ____

(i) $(V \sim \square) \sim \triangle =$ ____

(j) $V \sim (\square \sim \triangle) =$ ____

Does either table (b) or table (c) seem to represent an associative operation? Why? How could you prove your statement? What would another person have to do to prove you wrong?

Systems of Natural Numbers and Whole Numbers

The natural numbers form an algebraic system under both the operations of additions and multiplication. What are some of their properties? The natural numbers have an identity element with respect to multiplication. What is it? Do they have an identity element with respect to addition?

If we take the system of natural numbers and one more element, the number zero, we get a new and different mathematical system called the system of whole numbers. Which of the properties listed for the natural numbers are also possessed by the whole numbers? Do the whole numbers have any additional properties?

Exercises - 7

1. List the properties of these mathematical systems. How are they the same? In what ways are they different?
    - (a) The system of natural numbers and addition and multiplication.
    - (b) The system of whole numbers under addition and multiplication.
    - (c) The system whose set is the set of odd numbers and whose operation is multiplication.
    - (d) The system whose set is the set made up of zero and the multiples of 3, and whose operation is multiplication.
    - (e) The system whose set is the set made up of zero and the multiples of 3, and whose operation is addition.
    - (f) The system whose set is the even numbers and whose operation is addition.
    - (g) The system whose set is the fractions between 0 and 1 and whose operation is multiplication.
    - (h) The same set as in (g) under the operation of addition.
2. Make up an algebraic system (a combination of a set and an operation) of your own. Make at least a partial table for your system. (Could you make complete tables for the operations in Exercise 7 as we could for those in Exercise 6? Why?) List the properties of your system.

More about Modular Arithmetic

We have seen modular arithmetic for addition. If we put in multiplication we will get a different mathematical system. With both operations, modular arithmetic will be more like ordinary arithmetic.

Complete the multiplication tables for

mod 5 and mod 8

X	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	2
3	0	3	1	-	-
4	0	4	-	-	-

X	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	-	-	-	-
4	0	4	-	-	-	-	-	-
5	0	5	-	-	-	-	-	-
6	0	6	-	-	-	-	-	-
7	0	-	-	-	-	-	-	-

List the properties. (commutative, associative, closure, identity, inverse). Does the distributive property for multiplication over addition hold? Is it true that if a product is zero at least one of the factors is zero in mod 5? in mod 8?

Modular arithmetics may be thought of as mathematical systems with two operations. Just as we can solve problems using ordinary arithmetic, we can solve problems using modular arithmetic.

Exercises - 8

1. Find the sum of:

A	B
(a) 1 and 5 mod 8	(a) 3 and 2 mod 8
(b) 4 and 3 mod 8	(b) 6 and 5 mod 8
(c) 4 and 4 mod 8	(c) 7 and 7 mod 8
(d) 4 and 5 mod 8	(d) 0 and 2 mod 8
(e) 0 and 6 mod 8	(e) 7, 3, 5 and 1 mod 8
(f) 6 and 7 mod 8	(f) 6, 7, 7 and 5 mod 8

(g) 3, 5 and 2 mod 8

(h) 7, 6 and 4 mod 8

(i) 3, 7 and 6 mod 8

(j) 4 and 2 mod 5

(k) 7 and 2 mod 8

(l) 7 and 2 mod 9

(m) 7 and 2 mod 10

(n) the first three even numbers, mod 9 (0, 2 and 4)

(o) the multiples of three that are between 5 and 10 in arithmetic mod 12

(g) 5 and 4 mod 7

(h) 6, 3, 5 and 5 mod 7

(i) 10, 8, 6 mod 12

(j) 12, 3, 9, 2 mod 15

(k) the first four even numbers mod 12 (0, 2, 4 and 6)

(l) the first three prime numbers mod 9

(m) the multiples of three that are between 0 and 14 in arithmetic mod 15

(n) the greatest common factor of 4, 6 and 8, and the greatest common factor of 6 and 9 in arithmetic mod 12

(o) the numbers less than 10 in arithmetic mod 13.

2. Find the products:

(a) 3 x 5 mod 8

(b) 2 x 3 mod 8

(c) 2 x 3 mod 4

(d) 2 x 3 mod 5

(e) 2 x 3 mod 6

(f) 5 x 8 mod 7

(g) 3^2 mod 5

(h) 7^2 mod 8

(i) 6 x 4 mod 5

(j) 3 x 4 x 6 mod 9

(k) 4^3 mod 5

(a) 2 x 7 mod 8

(b) 5 x 3 mod 8

(c) 5 x 3 mod 9

(d) 5 x 3 mod 10

(e) 12 x 14 mod 18

(f) 6^2 mod 8

(g) 10^2 mod 12

(h) 7 x 6 x 7 mod 9

(i) 8 x 2 x (5 + 4) mod 11

(j) (3 + 4) x (9 - 5) x (5 - 2) mod 12

(k) 5^3 mod 9

3. Find the quotients:

Remember that division is always defined after we know about multiplication. Thus, in ordinary arithmetic the question "six divided by 2 is what?" means, really, "six is obtained by multiplying 2 by what?" An operation which begins with one of the numbers and the "answer" to another binary operation and asks for the other number is called an inverse operation. Division is the inverse operation of multiplication.

(a) $\frac{2}{3}$ mod 8
(b) $\frac{7}{6}$ mod 8
(c) $\frac{6}{2}$ mod 8
(d) $\frac{3}{4}$ mod 5
(e) $\frac{0}{2}$ mod 5
(f) $\frac{0}{2}$ mod 8

(a) $\frac{6}{5}$ mod 8
(b) $\frac{6}{2}$ mod 8
(c) $\frac{0}{4}$ mod 8
(d) $\frac{0}{4}$ mod 5
(e) $\frac{3}{2}$ mod 5
(f) $\frac{7}{3}$ mod 10

4. Compute: Remember, subtraction is the inverse operation to addition.

(a) 7 - 3 mod 8
(b) 3 - 7 mod 8
(c) 9 - 2 mod 10
(d) 2 - 9 mod 10
(e) 3 - 4 mod 5
(f) 3 - 4 mod 5
(g) 2 - 5 mod 9
(h) 3 - 6 mod 9
(i) 4 - 7 mod 9
(j) 4 - 8 mod 9
(k) Does (4-7)=(1-4) in arithmetic mod 8?
(l) Does (6-2)=(3-9) in arithmetic mod 10?

(a) 7 - 5 mod 8
(b) 5 - 7 mod 8
(c) 10 - 3 mod 11
(d) 3 - 10 mod 11
(e) 2 - 5 mod 6
(f) 2 - 5 mod 10
(g) 2 - 6 mod 12
(h) 3 - 7 mod 12
(i) 4 - 8 mod 12
(j) 4 - 9 mod 12
(k) For what modulus does 1 - 3 = 5?
(l) For what modulus does 5 - 9 = 10?

5. Study this modular arithmetic table:

x	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

(a) What is 5 x 5 mod 6? In this product the factors were the same. When a product has two identical factors we call one of them the square root of the product. 5 is the square root of 1 in arithmetic mod 6. Are any other examples like this listed in the table?

(b) Does 1 have any square roots other than 5?

(c) Does every number have two different square roots?

(d) Does any number have just one square root?

(e) Does any number have no square roots at all?

(f) Fill in this chart:

Number	Square Roots of the Number
0	
1	
2	
3	
4	
5	

6. Consider the system of natural numbers.

(a) Can you find a number that has a square root? What is it?

(b) Does more than one number have a square root?

(c) Does every number have a square root? Prove it.

(d) Does any natural number have more than one square root?

(e) Fill in the chart with names of natural numbers less than 110.

Number	1	4	9
Square Roots of the Number	1	2	3